ON IRREGULARITIES OF DISTRIBUTION II

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§ 1. Introduction

We shall be concerned with the distribution of points in the unit cube of \((k+1)\)-dimensional Euclidean space, where \(k \geq 1\). More precisely, let \(U_0 = [0, 1)\) and \(U_1 = (0, 1]\). Suppose we have a distribution \(\mathcal{P}\) of \(N\) points in \(U_0^{k+1}\), the unit cube consisting of points \(y = (y_1, \ldots, y_{k+1})\) with \(0 \leq y_i < 1\) \((i = 1, \ldots, k+1)\). For \(x = (x_1, \ldots, x_{k+1})\) in \(U_1^{k+1}\), let \(B(x)\) denote the box consisting of all \(y\) such that \(0 \leq y_i < x_i\) \((i = 1, \ldots, k+1)\), and let \(Z[\mathcal{P}; B(x)]\) denote the number of points of \(\mathcal{P}\) which lie in \(B(x)\). Let

\[
D[\mathcal{P}; B(x)] = Z[\mathcal{P}; B(x)] - Nx_1 \cdots x_{k+1}.
\]

We consider the \(L^w\)-norm

\[
\|D(\mathcal{P})\|_w = \left( \int_{U_0} \cdots \int_{U_0} |D(\mathcal{P}; B(x))|^w \, dx_1 \cdots dx_{k+1} \right)^{1/w}.
\]

We state two theorems which together essentially solve completely the problem of estimating \(\|D(\mathcal{P})\|_w\) as a function of \(N\) for \(W > 1\).

**Theorem 1** (Schmidt [14]). For every \(W > 1\), there exists a positive number \(c_1(k, W)\), depending only on \(k\) and \(W\), such that

\[
\|D(\mathcal{P})\|_w > c_1(k, W)(\log N)^{1/k}.
\]

The case \(W = 2\) of Theorem 1 was established much earlier by Roth [10]. Since \(\|D(\mathcal{P})\|_w\) is an increasing function of \(W\) for any fixed distribution \(\mathcal{P}\), Roth's result extends immediately to \(W \geq 2\).

**Theorem 2** (Chen [1]). Let \(W > 0\). For a suitable number \(c_2(k, W)\), depending only on \(k\) and \(W\), there exists, corresponding to every natural number \(N \geq 2\), a distribution \(\mathcal{P}\), which may depend on \(W\), of \(N\) points in \(U_0^{k+1}\) such that

\[
\|D(\mathcal{P})\|_w < c_2(k, W)(\log N)^{1/k}.
\]

The case \(W = 2\) of Theorem 2 was established by Roth [12]. However, special cases of Roth's result were obtained earlier by Davenport [4] \((k = 1)\) and Roth [11] \((k = 2)\). Clearly, their results extend immediately to \(W = 2\).

Much work was also done on the $L^\infty$-norm

$$\|D(\mathcal{P})\|_\infty = \sup_{x \in U_1^{k+1}} |D[\mathcal{P}; B(x)]|.$$  

As an immediate consequence of Theorem 1, we have

**Theorem 3A.** There exists a positive number $c_3(k)$, depending only on $k$, such that

$$\|D(\mathcal{P})\|_\infty > c_3(k)(\log N)^{4k}.$$  

This was improved in the case $k = 1$, the two-dimensional case.

**Theorem 3B (Schmidt [13]).** For every distribution $\mathcal{P}$ in $U_0^2$, we have

$$\|D(\mathcal{P})\|_\infty > (700)^{-1} \log N.$$  

An alternative proof of Theorem 3B (apart from the constant $(700)^{-1}$) was given recently by Halász [6], using an ingenious variation of Roth’s method in [10]. On the other hand, Theorem 3B is sharp, for we have

**Theorem 4 (Halton [7]).** For a suitable number $c_4(k)$, depending only on $k$, there exists, corresponding to every natural number $N \geq 2$, a distribution $\mathcal{P}$ of $N$ points in $U_0^{k+1}$ such that

$$\|D(\mathcal{P})\|_\infty < c_4(k)(\log N)^k.$$  

Halton’s proof of Theorem 4 is based on a generalization (cf. Hammersley [8]) of the method of van der Corput [3], and involves the expression of numbers $x \in [0, 1)$ in the form

$$x = \sum_{\nu=1}^{\infty} a_\nu p^{-\nu} \quad (0 \leq a_\nu < p),$$

using a different prime $p$ for each of the coordinates of $(x_1, \ldots, x_k)$ in $U_0^k$ to enable one to use the Chinese Remainder Theorem in the subsequent argument. Indeed, the proof of Theorem 2 by Roth and Chen is an adaptation of this approach.

Recently, Faure [5] showed that it is possible in certain cases to use the same prime $p$ (large compared to $k$) for every coordinate, so that one does not have to use the Chinese Remainder Theorem, which plays an important role in the original proofs of Theorems 2 and 4. In [5], Faure gave an alternative proof of Theorem 4.

In this paper, we shall show that the construction of Faure also leads to a proof of Theorem 2. Our method here is a variation of the method in [1] and [12], and is based on the consideration of “modifications” of a Faure sequence (or set) (see §§2–4) and taking an average of the “discrepancy” over a large number of such modifications. To show that this average is “small”, we make use of ideas in [1] and [12]. In [1] and
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[12], the Hammersley (Halton) sequences are constructed with the use of residue classes and the Chinese Remainder Theorem, and therefore have some nice "periodicity" properties. Unfortunately, Faure sequences do not have quite so nice periodicity properties. However, by the use of modifications of a Faure sequence (or set), we can make use of an idea which is implicit in the argument of [1] and [12].

The method here can also be applied to the Hammersley (Halton) sequences in [1] and [12], at the expense of some notational complication, to give a proof of Theorem 2. Indeed, the method would also work if sequences (or sets), which could be considered to be "intermediate cases" between Hammersley (Halton) sequences and Faure sequences (see § 2), could be shown to exist. In § 9, we prove some partial results to the contrary.

I am indebted to Professor Faure for sending me a manuscript of his paper [5].

For further discussion and references, see [2], [9] and [15].

§ 2. Hammersley (Halton) sequences and Faure sets

By an interval \( I \), we shall mean a half-open interval of the type \([\alpha_1, \alpha_2)\), while \( I_1 \times I_2 \) denotes the Cartesian product of \( I_1 \) and \( I_2 \). Throughout, we reserve the letter \( p \) (with or without suffices) for primes.

**Definition.** Let \( s \) be a non-negative integer. By an elementary \( p \)-type interval of order \( s \), we mean an interval of the type \( I = [\alpha_1, \alpha_2) \), contained in \( U_0 \), and where \( \alpha_1, \alpha_2 \) are consecutive integer multiples of \( p^{-s} \).

Let \( h \) be a non-negative integer, and let \( p_1, \ldots, p_k \) be primes, not necessarily distinct. Let \( q \) be a natural number.

**Definition.** By an elementary \( q \)-box of order \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^k \times [0, \infty) \), we mean a set of the form \( I_1 \times \cdots \times I_k \times I_0 \), where for each \( j = 1, \ldots, k \), \( I_j \) is an elementary \( p_j \)-type interval of order \( s_j (0 \leq s_j \leq h) \), and where \( I_0 \) is of the form \([\beta_1, \beta_2) \), with \( \beta_1, \beta_2 \) being consecutive non-negative integer multiples of \( q p_1^{s_1} \cdots p_k^{s_k} \).

It is clear that any elementary \( q \)-box in \( U_0^k \times [0, \infty) \) has "volume" \( q \).

We are interested in sets of points in \( U_0^k \times [0, \infty) \) such that many elementary boxes in \( U_0^k \times [0, \infty) \) contain the "right number of points". More precisely,

**Definition.** By a \( q \)-set of class \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^k \times [0, \infty) \), we mean an infinite set \( 2 \) of points in \( U_0^k \times [0, \infty) \) which has the property that every elementary \( q \)-box of order \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^k \times [0, \infty) \) contains exactly one point of \( 2 \).
The following two lemmas are obvious. In the case \( p_1, \ldots, p_k \) are all distinct, they are essentially Lemmas 1 and 2 in [1] respectively.

**Lemma 1.** Suppose \( \mathcal{D} \) is a \( q \)-set of class \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^b \times [0, \infty) \). Then for every \( s \) satisfying \( 0 \leq s \leq h \), \( \mathcal{D} \) is also a \( q \)-set of class \( s \) with respect to the same primes in \( U_0^b \times [0, \infty) \).

**Lemma 2.** Suppose \( \mathcal{D} \) is a \( 1 \)-set of class \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^b \times [0, \infty) \), and suppose \( I \) is an elementary \( p_j \)-type interval of order \( s \), with \( 1 \leq j \leq k \) and \( 0 \leq s \leq h \). Then the set

\[ \{(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_s, y) : x_t \in I \land (x_1, \ldots, x_s, y) \in \mathcal{D} \} \]

is a \( p_j \)-set of class \( h \) with respect to the primes \( p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k \) in \( U_0^{b+1} \times [0, \infty) \).

The existence of \( 1 \)-sets (and hence \( q \)-sets in general) in \( U_0^b \times [0, \infty) \) has been established in certain cases. We have the following theorems.

Let \( \mathbb{N} \) denote the set of all natural numbers.

**Theorem 5A** (Hammersley [8], Halton [7]). Suppose \( p_1, \ldots, p_k \) are distinct primes. For each \( n \in \mathbb{N} \) and each \( j = 1, \ldots, k \), write

\[ n-1 = \sum_{\nu=1}^{\infty} a_{j,\nu} p_\nu^{-1} \quad (0 \leq a_{j,\nu} < p), \]

where the integers \( a_{j,\nu} \) are uniquely determined by \( n \), and write

\[ x_j(n) = \sum_{\nu=1}^{\infty} a_{j,\nu} p_\nu^{-1}. \]

Then the set

\[ \{(x_1(n), \ldots, x_k(n), n-1) : n \in \mathbb{N} \} \quad (1) \]

is a \( 1 \)-set of class \( h \) with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^b \times [0, \infty) \) for any non-negative integer \( h \).

Suppose

\[ x = \sum_{\nu=1}^{\infty} a_{\nu} p^{-\nu} \quad (0 \leq a_{\nu} < p), \]

where all but a finite number of the integers \( a_{\nu} \) are zero. We define

\[ \mathcal{E} x = \sum_{\nu=1}^{\infty} b_{\nu} p^{-\nu}, \]

where \( 0 \leq b_{\nu} < p \) and

\[ b_{\nu} = \sum_{\mu=\nu}^{\infty} \binom{\mu-1}{\nu-1} a_{\mu} \pmod{p}. \]
For any integer \( m > 1 \), we let
\[
\mathcal{Q}^m x = \mathcal{Q}^{m-1} x.
\]

**Theorem 5B** (Faure [5]). Suppose \( p \) is a prime and \( p \geq k \). For each \( n \in \mathbb{N} \), write
\[
n - 1 = \sum_{v=1}^{\infty} a_v p^{v - 1} \quad \text{and} \quad x_n = \sum_{v=1}^{\infty} a_v p^{-v}.
\]
Then the set
\[
\{(x_n, \mathcal{Q}x_n, \ldots, \mathcal{Q}^{k-1}x_n, n - 1): \ n \in \mathbb{N}\}
\]
is a 1-set of class \( h \) with respect to the primes \( p, \ldots, p \) in \( U_0^k \times [0, \infty) \) for any non-negative integer \( h \).

The set (1) is usually known as a Hammersley (Halton) sequence, while we call the set (2) a Faure sequence. In this paper, \( q \)-sets with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^k \times [0, \infty) \) shall be called Hammersley (Halton) \( q \)-sets if \( p_1, \ldots, p_k \) are distinct, and Faure \( q \)-sets if all the primes are equal and at least \( k \).

We shall show that the restriction \( p \geq k \) on Faure \( q \)-sets is sharp.

**Theorem 6A.** Suppose \( p \) is a prime and \( p < k \). Then there are no 1-sets of class \( h \) with respect to the prime \( p \) in \( U_0^k \times [0, \infty) \) unless \( h = 0 \).

The special case \( p = 2 \) of Theorem 6A was proved by Sobol [16].

It is not known whether \( q \)-sets with respect to the primes \( p_1, \ldots, p_k \) in \( U_0^k \times [0, \infty) \) exist if the primes \( p_1, \ldots, p_k \) are not all distinct and not all equal. By Theorem 6A, it is clear that any prime \( p \) cannot be used more than \( p \) times. On the other hand, we can prove the following

**Theorem 6B.** There are no 1-sets of class \( h \) with respect to the primes 2, 2, 3 in \( U_0^3 \times [0, \infty) \) unless \( h \leq 1 \).

The following theorem is easier, and we omit the proof.

**Theorem 6C.** There are no 1-sets of class \( h \) with respect to the primes 2, 3, 3 in \( U_0^3 \times [0, \infty) \) unless \( h = 0 \).

We shall prove Theorems 6A and 6B in § 9.

§ 3. The discrepancy function \( E \)

In §§ 3–8, we shall prove Theorem 2 using Faure sets. We assume that \( p \) is a fixed prime at least equal to \( k \). We shall consider only Faure \( q \)-sets with respect to this prime \( p \), and shall sometimes omit reference to \( p \).

We begin by defining an appropriate discrepancy function for \( q \)-sets in \( U_0^k \times [0, \infty) \).
Suppose $B = I_1 \times \cdots \times I_k \times I^*$, where for $j = 1, \ldots, k$, $I_j \subset U_0$ and where $I^*$ is of the type $[0, Y)$, with $Y$ positive but otherwise unrestricted. For any infinite Faure $q$-set $\mathcal{F}_q$ in $U_0^h \times [0, \infty)$, we define $E[\mathcal{F}_q; B]$ by

$$E[\mathcal{F}_q; B] = Z[\mathcal{F}_q; B] - q^{-1}V(B),$$

where $Z[\mathcal{F}_q; B]$ denotes the number of points of $\mathcal{F}_q$ falling into $B$, and $V(B)$ denotes the volume of $B$. We note that $q^{-1}V(B)$ is the "expected number" of points of $\mathcal{F}_q$ falling into $B$. The analogues of (3) in lower-dimensional cases are defined in the same way. Note that if $B = B_1 \cup B_2$, where $B_1 \cap B_2 = \emptyset$, then

$$E[\mathcal{F}_q; B] = E[\mathcal{F}_q; B_1] + E[\mathcal{F}_q; B_2].$$

We are interested in boxes $B$ where $I_1, \ldots, I_k$ are unions of elementary $(p$-type) intervals.

**Definition.** Let $s$ be a non-negative integer. Suppose further that for each $j = 1, \ldots, k$, $I_j = [0, \eta_j)$, where $0 < \eta_j \equiv 1$ and $\eta_j$ is an integer multiple of $p^s$. Then we say that $B$ is a box of class $s$ in $U_0^h \times [0, \infty)$.

We reserve the symbol $B^*$ for such boxes.

§ 4. The matrix $A$

We denote by $A_{p,k}$ the set of all matrices $A = (a_{ij})$ with integer entries $a_{ij}$ satisfying $0 \leq a_{ij} < p$ for each $i = 1, \ldots, k$ and $t = 1, \ldots, h$. It is clear that $A_{p,k}$ has $p^{kh}$ elements.

Let $A = (a_{ij}) \in A_{p,k}$ and let $\mathcal{F}_q$ be a Faure $q$-set of class $h$ in $U_0^h \times [0, \infty)$. We construct the set $\mathcal{F}_q(A)$ as follows: Suppose $x = (x_1, \ldots, x_k, y) \in \mathcal{F}_q$. For $i = 1, \ldots, k$, write

$$x_i = \sum_{t=1}^{h} b_{it} p^{-t} + c_i,$$

where $b_{it}$ are integers satisfying $0 \leq b_{it} < p$, and where $0 \leq c_i < p^{-h}$. For $i = 1, \ldots, k$ and $t = 1, \ldots, h$, let $b_{it}^*$ be defined by

$$0 \leq b_{it}^* < p \quad \text{and} \quad b_{it}^* = b_{it} + a_{ii} \pmod{p},$$

and let

$$x_i^\wedge = \sum_{t=1}^{h} b_{it}^* p^{-t} + c_i.$$  (5)

Then clearly $x_i^\wedge \in [0, 1)$. Let

$$x^\wedge = (x_1^\wedge, \ldots, x_k^\wedge, y).$$

We define the set $\mathcal{F}_q(A)$ by

$$\mathcal{F}_q(A) = \{x^\wedge \in U_0^h \times [0, \infty): x \in \mathcal{F}_q\}. $$
It follows easily, by considering the effect of each entry of $A$ separately, that

**Lemma 3.** If $\mathcal{F}_q$ is a Faure $q$-set of class $h$ in $U_0^k \times [0, \infty)$, then $\mathcal{F}_q(A)$ is also a Faure $q$-set of class $h$ in $U_0^k \times [0, \infty)$.

On the other hand,

**Lemma 4.** Suppose $\mathcal{F}_q$ is a Faure $q$-set of class $h$ in $U_0^k \times [0, \infty)$, $A = (a_{ij}) \in \mathcal{A}_{k,h}$ and $B = I_1 \times \cdots \times I_k \times I^*$ is a box in $U_0^k \times [0, \infty)$, where for each $i = 1, \ldots, k$, $I_i$ is a union of elementary intervals of order $s$, where $0 \leq s \leq h$. Then the function $E[\mathcal{F}_q(A); B]$ is independent of the entries $a_{ij}$ of $A$ for every $i = 1, \ldots, k$ and $t = s + 1, \ldots, h$.

**Proof.** The lemma follows immediately, as, by (5), $x_i^h$ falls into an elementary interval of order $s$ determined uniquely by $b_1^h, \ldots, b_s^h$.

It is convenient to introduce the following notation. Let $A \in \mathcal{A}_{k,h}$. For $j = 1, \ldots, k$ and $s = 0, \ldots, h$, we write $A_{h,j}$ for the matrix obtained from $A$ by replacing $a_{i,k+1}, \ldots, a_{i,h}$ in $A$ by 0, and write $A_s$ for the matrix obtained from $A$ by replacing $a_{i,s+1}, \ldots, a_{i,h}$ in $A$ by 0 for every $i = 1, \ldots, k$. We see that there are $p^{(k-1)h+s}$ choices of $A_{h,s} \in \mathcal{A}_{k,h}$ and $p^h$ choices of $A_s \in \mathcal{A}_{k,h}$. Throughout,

$$\sum_{A \in \mathcal{A}_{k,h}} f(A) \quad \text{and} \quad \sum_{A_s \in \mathcal{A}_{k,h}} f(A_s)$$

denote respectively a sum over the $p^h$ choices of $A \in \mathcal{A}_{k,h}$ and a sum over the $p^h$ choices of $A_s \in \mathcal{A}_{k,h}$, while

$$\sum_{A_{h,s} \in \mathcal{A}_{k,h}} f(A_{h,s})$$

denotes a sum over the $p^{(k-1)h+s}$ choices of $A_{h,s} \in \mathcal{A}_{k,h}$.

§ 5. **Statement of the basic result and the key identity**

We state the following intermediate result from which Theorem 2 will be deduced in § 8. We also assume that $W$ is an even positive integer. In this section and the next two, we shall prove the

**Main Lemma.** For a suitable constant $C(p, k, W)$, depending only on $p$, $k$ and $W$, we have, for any Faure $q$-set $\mathcal{F}_q$ of class $h$ (with respect to the prime $p$) in $U_0^k \times [0, \infty)$, and, for any box $B^*$ of class $h$ in $U_0^k \times [0, \infty)$,

$$\sum_{A \in \mathcal{A}_{k,h}} E[\mathcal{F}_q(A); B^*]^W \leq C(p, k, W)p^{kh}h^kW.$$  

It is sufficient to prove the Main Lemma for $q = 1$. For if $\mathcal{F}'_q$ is the Faure 1-set of class $h$ in $U_0^k \times [0, \infty)$ containing precisely those points
\( x' = (x_1, \ldots, x_n, q^{-1}y) \), where \( x = (x_1, \ldots, x_n, y) \in \mathcal{F}_n \), and if \( \overline{B^*} = I_1 \times \cdots \times I_q \times [0, q^{-1}Y) \), where \( B^* = I_1 \times \cdots \times I_q \times [0, Y) \), then

\[
E[\mathcal{F}_n(A); B^*] = E[\mathcal{F}_n(A); \overline{B^*}]
\]

for every \( A \in \mathcal{A}_k \).

Let \( \mathcal{F} \) be any Faure 1-set of class \( h \) in \( U_n^* \times [0, \infty) \). We keep this set \( \mathcal{F} \) fixed throughout. Let \( A \in \mathcal{A}_k \).

Suppose

\[
B^* = B^*(\eta, Y) = [0, \eta_1) \times \cdots \times [0, \eta_k) \times I^*,
\]

where \( I^* = [0, Y) \), is a box of class \( h \) in \( U_n^* \times [0, \infty) \). We now partition the box \( B^* \) into a union of disjoint boxes as in [1]. (Take \( p = p_1 = \cdots = p_k \) in [1].) We also follow the notation in [1].) For \( j = 1, \ldots, k \) and \( s = 0, \ldots, h \), let \( \xi_{ja} \) denote the greatest integer multiple of \( p^{-1} \) not exceeding \( \eta_j \); further, for \( s \neq 0 \), write

\[
\nu_{ja} = p^s (\xi_{ja} - \xi_{ja-1}).
\]

Then \( \nu_{ja} \) is an integer and

\[
0 \leq \nu_{ja} < p
\]

for \( j = 1, \ldots, k \) and \( s = 1, \ldots, h \). For \( s = 0, \ldots, h \), write

\[
B_{ja}^* = [0, \xi_{ja}) \times \cdots \times [0, \xi_{ja}) \times I^*,
\]

so that \( B_{ja}^* \) is the largest box of class \( s \) in \( U_n^* \times [0, \infty) \) which is contained in \( B^* \). We now consider the complement of \( B_{ja}^* \) in \( B_{ja}^* \). For \( j = 1, \ldots, k \) and \( s = 1, \ldots, h \), let

\[
B_{ja} = [0, \xi_{ja-1}) \times \cdots \times [0, \xi_{j-1}a) \times [\xi_{ja-1}, \xi_{ja})
\]

\[
\times [0, \xi_{j+1}a) \times \cdots \times [0, \xi_{ja}) \times I^*.
\]

In other words, \( B_{ja} \) denotes the part of the complement of \( B_{ja}^* \) in \( B_{ja}^* \) which is contained in

\[
[\xi_{ja-1}, \xi_{ja}) \times [0, 1)^k \times I^*,
\]

\( B_{2a} \) denotes the part of the remainder which is contained in

\[
[0, 1) \times [\xi_{2a-1}, \xi_{2a}) \times [0, 1)^k \times I^*,
\]

and so on. It follows that for \( s = 1, \ldots, h \),

\[
B_{ja}^* = B_{ja-1}^* \cup B_{ja} \cup \cdots \cup B_{ja},
\]

and that the union is pairwise disjoint, so that from (4) and (10),

\[
E[\mathcal{F}(A); B_{ja}^*] = E[\mathcal{F}(A); B_{ja-1}] + \sum_{I \neq I^*} E[\mathcal{F}(A); B_{ja}].
\]

For \( j = 1, \ldots, k \) and \( s = 1, \ldots, h \), the interval \([\xi_{ja-1}, \xi_{ja})\) is a union of exactly \( \nu_{ja} \) disjoint elementary intervals of order \( s \). We denote them by
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Then for \( j=1, \ldots, k \) and \( s=1, \ldots, h \),

\[
B_{j,s} = \bigcup_{\alpha=1}^{v_j} B_{j,s,\alpha},
\]

and the union is pairwise disjoint, so that by (4),

\[
E[\mathcal{F}(A); B_{j,s}] = \sum_{\alpha=1}^{v_j} E[\mathcal{F}(A); B_{j,s,\alpha}].
\] (13)

For \( j=1, \ldots, k \) and \( s=1, \ldots, h-1 \), let

\[
\tilde{J}_{j,s} = [\xi_{j,s}, \xi_{j,s} + p^{-1}),
\]

so that \( \tilde{J}_{j,s} \) is the elementary interval of order \( s \) containing the complement of \([0, \xi_{j,s}) \) in \([0, \eta_j)\). We also let

\[
\tilde{B}_{j,s} = [0, \xi_{j,h-1}) \times \cdots \times [0, \xi_{j,1,h-1}) \times \tilde{J}_{j,s}
\]

\[
\times [0, \xi_{j,1,h}) \times \cdots \times [0, \xi_{j,h}) \times I^*.
\] (15)

so that \( \tilde{B}_{j,s} \) is obtained from \( B_{j,h} \) (see (9) with \( s=h \)) by replacing \([\xi_{j,h-1}, \xi_{j,h}) \) by \( \tilde{J}_{j,s} \). Note that \( B_{j,h} \) has \( h-1 \) modifications, namely \( B_{j,1}, \ldots, B_{j,h-1} \).

Our next lemma is the analogue of Lemma 4 in [1]. However, its proof is different, although the idea is implicit in [1]. Also, it is the motivation for introducing the matrices \( A \in \mathcal{A}_{k,h} \).

**Lemma 5.** For \( j=1, \ldots, k \) and \( \alpha=1, \ldots, v_{j,h} \), we have

\[
\sum_{\alpha=0}^{v_{j,h}-1} E[\mathcal{F}(A); B_{j,h,\alpha}] = E[\mathcal{F}(A_{j,h-1}); \tilde{B}_{j,h-1}].
\] (16)

Also, for \( j=1, \ldots, k \) and \( s=1, \ldots, h-2 \), we have

\[
\sum_{\alpha=0}^{p-1} E[\mathcal{F}(A_{j,s+1}); \tilde{B}_{j,s+1}] = E[\mathcal{F}(A_{j,s}); \tilde{B}_{j,s}].
\] (17)

**Proof.** We first prove (16). Note first of all from (12) with \( s=h \) and (15) that the only difference between \( B_{j,h,\alpha} \) and \( \tilde{B}_{j,h-1} \) is that \( J_{j,h,\alpha} \), an elementary interval of order \( h \), in \( B_{j,h,\alpha} \) is replaced by \( \tilde{J}_{j,h-1} \), the elementary interval of order \( h-1 \) containing \( J_{j,h,\alpha} \). By (5), if \( x_{j}^{\alpha} \in \tilde{J}_{j,h-1} \), then \( a_{j,1}, \ldots, a_{j,h-1} \) are uniquely determined. Also, for exactly one value of \( a_{j,\alpha} \), we also have \( x_{j}^{\alpha} \in J_{j,h,\alpha} \). (16) follows immediately, on noting (3) and observing that \( pV(B_{j,h,\alpha}) = V(\tilde{B}_{j,h-1}) \). The proof of (17) is similar.
Combining (13) with \( s = h \), (16) and (17), we have that for \( j = 1, \ldots, k \) and \( s = 1, \ldots, h-1 \).

\[
\sum_{a_{i,s}=0}^{r-1} \cdots \sum_{a_{0,s}=0}^{r-1} E[\mathcal{F}(A); B_{j,s}] = v_{j,h} E[\mathcal{F}(A_{j,s}); \bar{B}_{j,s}].
\]

For \( s = 1, \ldots, h \), write, for the sake of simplicity,

\[
G_s(A) = \sum_{j=1}^{k} E[\mathcal{F}(A); B_{j,s}].
\]

Then by (4), (9) and Lemma 4,

\[
G_s(A) = G_s(A_s).
\]

Let

\[
T_0 = \sum_{A_{h-1} \in \mathcal{A}_h} E[\mathcal{F}(A_{h-1}); B_{h-1}^{*}]^w,
\]

and

\[
T_1 = \sum_{A_{h-1} \in \mathcal{A}_h} E[\mathcal{F}(A); B_{s}^{*}]^{w-1} G_s(A).
\]

For \( w = 2, \ldots, W \), let

\[
T_w = \sum_{A_{h-1} \in \mathcal{A}_h} E[\mathcal{F}(A_{h-1}); B_{s}^{*}]^{w} G_s(A).
\]

Further, for \( j = 1, \ldots, k \) and \( s = 1, \ldots, h-1 \) and \( w = 1, \ldots, W-1 \), write

\[
S_{j,h,w} = \sum_{A_{h-1} \in \mathcal{A}_h} E[\mathcal{F}(A_{h-1}); B_{s}^{*}]^{w-1} G_s^{*}(A) E[\mathcal{F}(A_{j,s}); \bar{B}_{j,s}].
\]

The next lemma is the key lemma in the proof of Theorem 2, and is the analogue of Lemma 5 in [1].

**Lemma 6.** We have, for \( h \geq 2 \), that

\[
\sum_{A \in \mathcal{A}_h} E[\mathcal{F}(A); B_s^*]^w = p^k T_0 + WT_1 + \sum_{w=2}^{W} \binom{W}{w} T_w
\]

\[
+ W \sum_{j=1}^{k} v_{j,h} \sum_{s=1}^{h-1} \sum_{w=1}^{w-1} \binom{w-1}{w} S_{j,h,w}.
\]

**Proof.** Note, first of all, that for every \( A \in \mathcal{A}_h \) and \( s = 1, \ldots, h \), we have, by Lemma 4,

\[
E[\mathcal{F}(A); B_{s-1}^*] = E[\mathcal{F}(A_{s-1}); B_{s-1}^*].
\]

By (6) and (8), we have \( B_s^* = B_h^* \). By taking \( W \)th powers on both sides of (11) with \( s = h \), and using binomial expansion on the right-hand side, we
have, in view of (19) and (26), that
\[
E[\mathcal{F}(A); B^*]^w = E[\mathcal{F}(A_{h-1}); B^*_{h-1}]^w
+ \sum_{w=2}^{w} \binom{w}{w-1} E[\mathcal{F}(A_{h-1}); B^*_{h-1}]^{w-1} G^*_v(A)
+ \text{WE}[\mathcal{F}(A); B^*_{h-1}]^{w-1} \sum_{j=1}^{k} E[\mathcal{F}(A); B_{j,h}].
\]
By (19), (21), (22) and (23), we see that to prove (25), it remains to show that for \(j = 1, \ldots, k,\)
\[
\sum_{A \in \mathcal{A}_{k,h}} E[\mathcal{F}(A); B^*_{h-1}]^{w-1} E[\mathcal{F}(A); B_{j,h}]
= \sum_{A \in \mathcal{A}_{k,h}} E[\mathcal{F}(A); B^*_{h-1}]^{w-1} E[\mathcal{F}(A); B_{j,h}] + \nu_{j,h} \sum_{s=1}^{h-1} \sum_{w=1}^{w-1} \binom{w-1}{w} S_{j,s,w}.
\]
By repeated application of (11) and the binomial theorem, we see that for each \(A \in \mathcal{A}_{k,h},\)
\[
E[\mathcal{F}(A); B^*_{h-1}]^{w-1} = E[\mathcal{F}(A); B^*_{h-1}]^{w-1}
+ \sum_{s=1}^{h-1} \sum_{w=1}^{w-1} \binom{w-1}{w} E[\mathcal{F}(A); B^*_{h-1}]^{w-w-1} G^*_v(A),
\]
so that it suffices to show that for every \(j = 1, \ldots, k\) and \(s = 1, \ldots, h-1\) and \(w = 1, \ldots, W-1,\)
\[
\sum_{A \in \mathcal{A}_{k,h}} E[\mathcal{F}(A); B^*_{h-1}]^{w-1} G^*_v(A) E[\mathcal{F}(A); B_{j,h}] = \nu_{j,h} S_{j,s,w}. \tag{27}
\]
Clearly, (27) follows from (26), (20), (18) and (24), and this completes the proof of Lemma 6.

§ 6. The case \(k = 1\)

The following lemma is only applicable for the case \(k = 1.\)

**Lemma 7.** Let \(A \in \mathcal{A}_{1,h}.\) For \(s = 1, \ldots, h,\)
\[
|G_s(A)| = |E[\mathcal{F}(A); B_{1,s}]| < p; \tag{28}
\]
also, for \(s = 1, \ldots, h-1,\)
\[
|E[\mathcal{F}(A); B_{1,s}]| \leq 1. \tag{29}
\]

*Proof.* By (7) and (13) with \(j = 1,\) we see that to prove (28), it suffices to show that for every \(\alpha = 1, \ldots, \nu_{1,s},\)
\[
|E[\mathcal{F}(A); B_{1,s,\alpha}]| \leq 1. \tag{30}
\]
Recall (12) with \(j = k = 1.\) We see that \(B_{1,s,\alpha} = J_{1,s,\alpha} \times I^*,\) where for every
\[ \alpha = 1, \ldots, \nu_{1,s}, J_{1,\alpha} \] is an elementary interval of order \( s \). Let \( I^* = [0, Y) \), and let \( Y_0 \) be the greatest integer multiple of \( p^* \) not exceeding \( Y \). Then by (4), we have that
\[ E[\mathcal{F}(A); B_{1,\alpha}] = E[\mathcal{F}(A); J_{1,\alpha} \times [0, Y_0)] + E[\mathcal{F}(A); J_{1,\alpha} \times [Y_0, Y)]. \tag{31} \]

We now note that \( J_{1,\alpha} \times [0, Y_0) \) is a union of \( p^* \) disjoint elementary 1-boxes of order \( h \) (with respect to the prime \( p \)) in \( U_0 \times [0, \infty) \). Since \( \mathcal{F}(A) \) is a 1-set of class \( h \) in \( U_0 \times [0, \infty) \), we have \( \mathcal{Z}[\mathcal{F}(A); J_{1,\alpha} \times [0, Y_0)] = p^* Y_0 \). On the other hand, \( V(J_{1,\alpha} \times [0, Y_0)) = p^* Y_0 \), and so by (3),
\[ E[\mathcal{F}(A); J_{1,\alpha} \times [0, Y_0)] = 0. \tag{32} \]

On the other hand, \( J_{1,\alpha} \times [Y_0, Y) \) is contained in \( J_{1,\alpha} \times [Y_0, Y_0+p^*) \), where the latter is an elementary 1-box of order \( h \) in \( U_0 \times [0, \infty) \). So \( 0 \leq \mathcal{Z}[\mathcal{F}(A); J_{1,\alpha} \times [Y_0, Y)] \leq 1 \). Furthermore, \( 0 \leq V(J_{1,\alpha} \times [Y_0, Y]) < 1 \). Hence by (3),
\[ |E[\mathcal{F}(A); J_{1,\alpha} \times [Y_0, Y)]| \leq 1. \tag{33} \]

(30) follows on combining (31), (32) and (33). This completes the proof of (28). The proof of (29) is similar to the proof of (30), since by (14) and (15), \( B_{1,s} = J_{1,s} \times I^* \), where \( J_{1,s} \) is an elementary interval of order \( s \).

We shall prove by induction on \( h \) that the Main Lemma for \( k = 1 \) holds for a constant \( C \) satisfying
\[ C = (2^{w+1} p)^w. \tag{34} \]

Note that in particular,
\[ C > p^w. \tag{35} \]

Suppose first that \( B^* = I_1 \times I^* \), where \( I_1 = [0, 1) \). Then by (4), writing \( I^* = [0, Y) \),
\[ E[\mathcal{F}(A); B^*] = E[\mathcal{F}(A); I_1 \times [0, [Y])] + E[\mathcal{F}(A); I_1 \times [[Y], Y])] \]
for every \( A \in \mathcal{A}_{1,h} \). Then it is clear (see the proof of (32)) that
\[ E[\mathcal{F}(A); I_1 \times [0, [Y])] = 0 \]
and (see the proof of (33)) that
\[ |E[\mathcal{F}(A); I_1 \times [[Y], Y])| \leq 1. \]

Hence the Main Lemma for \( k = 1 \) is proved if \( I_1 = [0, 1) \). We therefore suppose that \( I_1 \neq [0, 1) \), so that by (8), \( B_0^* = \emptyset \). For \( h = 1 \), we know, by (10) and (28), that
\[ |E[\mathcal{F}(A); B^*]| = |E[\mathcal{F}(A); B_1^*]| = |E[\mathcal{F}(A); B_{1,1}]| < p, \]
and so, in view of (35), the Main Lemma for \( k = 1 \) holds for \( h = 1 \).
Suppose now that $h > 1$ and that the Main Lemma for $k = 1$ holds when $h$ is replaced by any smaller positive integer, so that, in particular, for any $s = 2, \ldots, h$, we have, in view of Lemma 4,

$$\sum_{A \in \mathcal{A}_s} E[\mathcal{F}(A_{s-1}); B_{s-1}^*] = p^{h-s} \sum_{A \in \mathcal{A}_s} E[\mathcal{F}(A_{s-1}); B_{s-1}^*] \leq Cp^h(s - 1)^{hw}.$$  \hfill (36)

By (21) and (36),

$$pT_0 \leq Cp^h(h - 1)w.$$ \hfill (37)

On the other hand, since $B_0^* = \emptyset$, we have $E[\mathcal{F}(A); B_0^*] = 0$, and so by (22),

$$T_1 = 0.$$ \hfill (38)

Applying Hölder's inequality, (36) and (28) to (23), we have, for $w = 2, \ldots, W$,

$$T_w \leq C^{(w-w)/w}p^{w + h(h - 1)(w-w)} \leq C^{1 - 1/w}p^{h + 1}(h - 1)^{w-1},$$

in view of (35), so that

$$\sum_{w=2}^{W} \left(\frac{W}{w}\right) T_w \leq C^{1 - 1/w}2wp^{h+1}(h - 1)^{w-1}.$$ \hfill (39)

Applying Hölder's inequality, (28), (29) and (36) to (24), we have, for $s = 2, \ldots, h - 1$ and $w = 1, \ldots, W - 1$, in view of (35), that

$$S_{1,s,w} \leq C^{(w-w-1)/w}p^{w/s}p^w(s - 1)^{w-1} \leq C^{1 - 1/w}p^s(h - 1)^{w-1}.$$  \hfill (39)

Furthermore,

$$S_{1,1,w-1} \leq p^{w-1-s} \leq C^{1 - 1/w}p^s(h - 1)^{w-1}.$$  \hfill (40)

For $w = 1, \ldots, W - 2$, we have $S_{1,1,w} = 0$. It follows, in view of (7) and the above, that

$$W_{s-1} \sum_{s=1}^{h-1} \sum_{w=1}^{W-1} \left(\frac{W}{w}\right) S_{1,s,w} \leq C^{1 - 1/w}p^{h+s}2^{w-1}W(h - 1)^{w-1}.$$ \hfill (40)

Combining (25), (37), (38), (39) and (40), we have

$$\sum_{A \in \mathcal{A}_s} E[\mathcal{F}(A); B_{s}^*]^{w} \leq Cp^{h(h - 1)^{w} + C^{-1/w}(2^{w-2}Wp(h - 1)^{w-1})} \leq Cp^{h(h - 1)^{w} + C^{-1/w}Wp(h - 1)^{w-1}} \leq Cp^{h(h - 1)^{w} + \frac{1}{2}W(h - 1)^{w-1}} \leq Cp^{h^{1/w}},$$

in view of (34). The proof of the Main Lemma for $k = 1$ is now complete.
§ 7. The case \( k \geq 2 \)

We suppose now that the corresponding Main Lemma in \( U_0^{k-1} \times [0, \infty) \)
holds with constant \( C_0 \) (assumed to be greater than 1) in place of \( C \). We shall prove by induction on \( h \) that the Main Lemma holds for a constant \( C \) satisfying

\[
C = C(p, k, W) = 2^{kW + W} p^{k(W + 1)} C_0. \tag{41}
\]

Clearly

\[
C > p^{k(W + 1)} C_0 \tag{42}
\]

and

\[
C > p^{kW}. \tag{43}
\]

We first establish a link which enables us to use the analogue of the
Main Lemma in \( U_0^{k-1} \times [0, \infty) \).

Let \( A = (a_i) \in \mathfrak{A}_{k,h} \). We write \( a = (a_{i1}, \ldots, a_{ih}) \), and write \( A^j \in \mathfrak{A}_{k-1,h} \)
for the matrix obtained from \( A \) by removing the \( j \)th row. For \( j = 1, \ldots, k \)
and \( s = 1, \ldots, h \) and \( \alpha = 1, \ldots, \eta_{j,s} \), we define the set \( \mathcal{F}_{a,j,s,\alpha}(A^j) \) of points
in \( U_0^{k-1} \times [0, \infty) \) by

\[
\mathcal{F}_{a,j,s,\alpha}(A^j) = \{ (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_h, y) : x_i \in J_{j,s,\alpha} \cap (x_1, \ldots, x_h, y) \in \mathcal{F}(A) \}. \]

Then by Lemma 2, \( \mathcal{F}_{a,j,s,\alpha}(A^j) \) is a Faure \( p^s \)-set of class \( h \) in \( U_0^{k-1} \times [0, \infty) \). On the other hand, let \( B^*_j \) be defined for the above values of \( j \) and \( s \) by

\[
B^*_j = [0, \xi_{j,s-1}] \times \cdots \times [0, \xi_{j-1,s-1}] \times [0, \xi_{j+1,s}] \times \cdots \times [0, \xi_{k,s}] \times I^*. \]

Then \( B^*_j \) is a box of class \( s \) in \( U_0^{k-1} \times [0, \infty) \). It is easy to see that for the
above values of \( j \), \( s \) and \( \alpha \), we have

\[
E[\mathcal{F}(A); B^*_j] = E[\mathcal{F}_{a,j,s,\alpha}(A^j); B^*_j]. \tag{44}
\]

It follows, on our assumption of the Main Lemma in \( U_0^{k-1} \times [0, \infty) \), and in view of (44) and Lemma 4, that for every \( j = 1, \ldots, k \) and \( s = 1, \ldots, h \)
and \( \alpha = 1, \ldots, \eta_{j,s} \),

\[
\sum_{A \in \mathfrak{A}_{k-1}} E[\mathcal{F}(A); B^*_j]^W = \sum_{A \in \mathfrak{A}_{k-1}} E[\mathcal{F}_{a,j,s,\alpha}(A); B^*_j]^W, \tag{45}
\]

where the summation \( \sum \) denotes a sum over the \( p^s \) choices of \( a = (a_{i1}, \ldots, a_{ih}, 0, \ldots, 0) \). For each of the above choices of \( a \), we have

\[
\sum_{A \in \mathfrak{A}_{k-1}} E[\mathcal{F}_{a,j,s,\alpha}(A); B^*_j]^W \leq C_0 p^{(k-1)\xi(k-1)} W. \tag{46}
\]

On combining (45) and (46), we see that

\[
\sum_{A \in \mathfrak{A}_{k-1}} E[\mathcal{F}(A); B^*_j]^W \leq C_0 p^{kW(k-1)W}. \tag{47}
\]
It is trivial that for \( j = 1, \ldots, k \) and \( s = 1, \ldots, h \),
\[
\sum_{\Lambda_{j,s} \in d_{k,h}} G_{x}^{w}(A_{s}) = p^{(k-1)(h-s)} \sum_{\Lambda_{j,s} \in d_{k,h}} G_{x}^{w}(A_{s}). \tag{48}
\]
On the other hand, by (19) and (13), we have, for \( s = 1, \ldots, h \),
\[
\sum_{\Lambda_{j,s} \in d_{k,h}} G_{y}^{w}(A_{s}) = \sum_{\Lambda_{j,s} \in d_{k,h}} \left( \sum_{\alpha = 1}^{v_{k,s}} \sum_{\beta = 1}^{v_{k,s}} E[\mathcal{F}(A_{s}) ; B_{\Lambda_{j,s}}] \right)^{w}
\leq \left( \sum_{\Lambda_{j,s} \in d_{k,h}} \sum_{\alpha = 1}^{v_{k,s}} \sum_{\beta = 1}^{v_{k,s}} E[\mathcal{F}(A_{s}) ; B_{\Lambda_{j,s}}] \right)^{w}. \tag{49}
\]
On combining (47), (48) and (49), we have, for \( s = 1, \ldots, h \),
\[
\sum_{\Lambda_{j,s} \in d_{k,h}} G_{y}^{w}(A_{s}) \leq C_{0}p^{(k-1)h+s} p^{w(k+1)} e^{h(k-1)w}. \tag{50}
\]
Similarly, for \( j = 1, \ldots, k \) and \( s = 1, \ldots, h-1 \), on relating \( E[\mathcal{F}(A_{s}) ; B_{\Lambda_{j,s}}] \) to the discrepancy of certain \( p^{*} \)-set in the box \( B_{\Lambda_{j,s}}^{w} \), we have
\[
\sum_{\Lambda_{j,s} \in d_{k,h}} E[\mathcal{F}(A_{s}) ; B_{\Lambda_{j,s}}]^{w} \leq C_{0}p^{(k-1)h+s} h^{k(k-1)w}. \tag{51}
\]
We may assume that \( B^{*} = I_{1} \times \cdots \times I_{k} \times I^{*} \), where for some \( j = 1, \ldots, k \), \( I_{j} \neq [0,1) \). Otherwise, we have \( |E[\mathcal{F}(A) ; B^{*}]| \leq 1 \) trivially, and the Main Lemma follows. By (8), we see that \( B_{h}^{w} = \emptyset \). For \( h = 1 \), let \( I^{*} = [0, Y) \), and let \( Y_{0} \) be the greatest integer multiple of \( p^{k} \) not exceeding \( Y \). Then if \( B^{*} = I_{1} \times \cdots \times I_{k} \times I^{*} \), let \( B_{1} \) and \( B_{2} \) be defined by \( B_{1} = I_{1} \times \cdots \times I_{k} \times [0, Y_{0}) \) and \( B_{2} = I_{1} \times \cdots \times I_{k} \times [Y_{0}, Y) \). Then it is clear that \( E[\mathcal{F}(A) ; B_{1}] = 0 \). On the other hand, since \( B_{2} \subset I_{1} \times \cdots \times I_{k} \times [Y_{0}, Y_{0}+p^{k}] \), a union of at most \( p^{k} \) elementary 1-boxes of order 1 in \( U_{k}^{w} \times [0, \infty) \), we have that \( Z[\mathcal{F}(A) ; B_{2}] \leq p^{k} \). Clearly, \( V(B_{2}) < p^{k} \). Hence by (3), \( |E[\mathcal{F}(A) ; B_{2}]| \leq p^{k} \). It follows by (4) that \( |E[\mathcal{F}(A) ; B^{*}]| \leq p^{k} \), and so in view of (43), the Main Lemma holds for \( h = 1 \) with \( C \) defined by (41).

Suppose now that \( h > 1 \) and that the Main Lemma holds when \( h \) is replaced by any smaller positive integer, so that, in particular, for \( s = 2, \ldots, h \) and \( j = 1, \ldots, k \), in view of Lemma 4,
\[
\sum_{\Lambda_{j,s} \in d_{k,h}} E[\mathcal{F}(A_{s-1}) ; B_{\Lambda_{j,s}}^{w}]^{w} = p^{h-1} \sum_{\Lambda_{j,s} \in d_{k,h}} E[\mathcal{F}(A_{s-1}) ; B_{\Lambda_{j,s}}^{w}]^{w}
\leq C_{0} p^{h(k-1)w} (s-1)^{kw}. \tag{52}
\]
By (21) and (52),
\[
p^{h} T_{0} \leq C_{0} p^{h(k-1)w} (h-1)^{kw}. \tag{53}
\]
On the other hand, since \( B_{h}^{*} = \emptyset \), we have \( E[\mathcal{F}(A) ; B_{h}^{*}] = 0 \), and so by (22),
\[
T_{h} = 0. \tag{54}
\]
Applying Hölder's inequality, (52) and (50) with \( s = h \) (note that \( A_{l,h} = A_0 \)) to (23), we have, for \( w = 2, \ldots, W \),

\[
T_w \leq C^{(W-w)/W}C_0^{w/W}p^{kh}p^{(W+1)/W}(h-1)^{h}(W-w)h^{h}2kW(h-1)^{h-kW-1},
\]

in view of (42), so that

\[
\sum_{w=2}^{W} \left( \frac{W}{w} \right) T_w \leq C^{1-1/W}C_0^{1/W}p^{kh}p^{(W+1)/W}2^{kW}(h-1)^{h-kW-1}.
\]

Applying Hölder's inequality, (52), (50) and (51) to (24), we have, for \( j = 1, \ldots, k \) and \( s = 2, \ldots, h-1 \) and \( w = 1, \ldots, W-1 \), in view of (42),

\[
S_{j,s,w} \leq C^{(W-w-1)/W}C_0^{(W+1)/W}p^{(k-1)h-1}h^{h}h^{(W-1)/W}(h-1)^{h-kW-1}.
\]

Furthermore, for \( j = 1, \ldots, k \),

\[
S_{j,1,w-1} \leq C_0^{(k-1)h+1}p^{(W-1)/W}h^{(k-1)}(h-1)^{h-kW-1}.
\]

For \( j = 1, \ldots, k \) and \( w = 1, \ldots, W-2 \), we have \( S_{j,1,w} = 0 \). It follows, in view of (7) and the above, that

\[
W \sum_{j=1}^{W-1} \sum_{w=1}^{W-1} \left( \frac{W-1}{w} \right) S_{j,w,s} \leq C^{1-1/W}C_0^{1/W}p^{kh}p^{2^{h-kW-1}}W(h-1)^{h-kW-1} = C^{1-1/W}C_0^{1/W}p^{kh}2^{h-kW-1}(h-1)^{h-kW-1},
\]

since \( 2^{h-kW-1}W = 2^{h-kW-2} \leq 2^{kW} \). Combining (25), (53), (54), (55) and (56), we have

\[
\sum_{\omega} E[\mathcal{F}(A); B^{\omega}] \leq C^{p^{kh}}(h-1)^{h+kW} + 2C^{1-1/W}C_0^{1/W}p^{(W+1)/W}2^{kW}(h-1)^{h-kW-1} \leq C^{p^{kh}}(h-1)^{h+kW} + (h-1)^{h-kW-1} = C^{p^{kh}}h^{h+kW},
\]

in view of (41). The proof of the Main Lemma is now complete.

§ 8. Proof of Theorem 2

We follow [1] almost verbatim.

Let \( p \) be the smallest prime satisfying \( p \geq k \). For any natural number
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\( N \geq 2, \) let \( h \) be the integer satisfying
\[ p^{h-1} < N \leq p^h. \]

(57)

For any \( \theta = (\theta_1, \ldots, \theta_k) \) in \( U_k^1 \), and, for any real \( Y \) satisfying
\[ 0 < Y \leq N, \]

(58)

let \( B(\theta, Y) \) be the box defined by \( B(\theta, Y) = [0, \theta_1] \times \cdots \times [0, \theta_k] \times [0, Y] \).

Let \( \eta = \eta(\theta) = (\eta_1, \ldots, \eta_k) \) be defined such that, for every \( j = 1, \ldots, k \),
\[ \eta_j = \eta_j(\theta_j) = p^{-h} [-p^h \theta_j], \]
i.e. \( \eta \) is the least integer multiple of \( p^h \) not less than \( \theta_j \).

Let \( S \) be any fixed Faure 1-set of class \( h \) in \( U_k^0 \times [0, \infty) \), and let \( A \in \mathfrak{A}_{k,h} \).

**Lemma 8.** For any \( \theta \) in \( U_k^1 \) and any \( Y \) satisfying (58),
\[ |E[\Phi(A); B(\theta, Y)] - E[\Phi(A); B^*(\eta(\theta), Y)]| \leq k. \]

The proof of Lemma 8 is essentially the same as the proof of Lemma 7 in [1]. The complement of \( B(\theta, Y) \) in \( B^*(\eta(\theta), Y) \) is contained in a union of at most \( k \) elementary 1-boxes of order \( h \) in \( U_k^0 \times [0, \infty) \).

For any even positive integer \( W \), we know, by Lemma 8, that for any \( \theta \) in \( U_k^1 \), for any real \( Y \) satisfying (58), and for any \( A \in \mathfrak{A}_{k,h} \),
\[ E[\Phi(A); B(\theta, Y)]^\leq 2^W E[\Phi(A); B^*(\eta(\theta), Y)]^\leq (2k)^W. \]

(59)

By the Main Lemma, we have
\[ \sum_{A \in \mathfrak{A}_{k,h}} \int_{U_k} \int_0^N E[\Phi(A); B(\eta(\theta), Y)]^\leq d\theta_1 \cdots d\theta_k dY \leq k^W p^{kh} h^{2^W N}. \]

(60)

Combining (59) and (60), we see that there is a matrix \( A^* \in \mathfrak{A}_{k,h} \) such that
\[ \int_{U_k} \int_0^N E[\Phi(A^*); B(\theta, Y)]^\leq d\theta_1 \cdots d\theta_k dY \leq k^W (\log N)^{h^2 W N}, \]

(61)

in view of (57). Note that the choice of \( A^* \) may depend on the set \( \Phi \) and the value of \( W \). On recalling the definition of \( \Phi \), we see that the set
\[ \Phi = \{(x_1, \ldots, x_k, N^{-1} y): (x_1, \ldots, x_k, y) \in \Phi(A^*) \land 0 \leq y < N \} \]
gives a proof of Theorem 2 for even positive integers \( W \).

The case for general positive \( W \) follows immediately.
§ 9. Non-existence of 1-sets with respect to certain primes

In this section, we prove Theorems 6A and 6B. However, before we do so, it is useful to note the following generalization of the idea of Lemma 3.

Suppose $\mathcal{E}$ is a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_0^h \times [0, \infty)$. For $i = 1, \ldots, k$ and $t = 1, \ldots, h$, let $\theta_{it}$ be a permutation of the numbers $0, 1, \ldots, p_t - 1$, and let $\Theta = (\theta_{it})$. Suppose that $x = (x_1, \ldots, x_k, y) \in \mathcal{E}$. For $i = 1, \ldots, k$, write

$$x_i = \sum_{t=1}^{h} b_{it} p_t^{-t} + c_i,$$

where $b_{it}$ are integers satisfying $0 \leq b_{it} < p_t$, and where $0 \leq c_i < p_t^{-h}$. For $i = 1, \ldots, k$ and $t = 1, \ldots, h$, let $b_{it}^o$ be defined by

$$b_{it}^o = \theta_{it}(b_{it}),$$

and let

$$x_i^o = \sum_{t=1}^{h} b_{it}^o p_t^{-t} + c_i.$$

Then clearly $x_i^o \in [0, 1)$. Let

$$x^o = (x_1^o, \ldots, x_k^o, y).$$

We define the set $\mathcal{E}(\Theta)$ by

$$\mathcal{E}(\Theta) = \{x^o \in U_0^k \times [0, \infty) : x \in \mathcal{E}\}.$$

It follows easily, by considering the effect of each $\theta_{it}$ separately, that

LEMMA 9. If $\mathcal{E}$ is a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_0^h \times [0, \infty)$, then $\mathcal{E}(\Theta)$ is also a 1-set of class $h$ with respect to the same primes in $U_0^k \times [0, \infty)$.

Suppose $\mathcal{E}$ is a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_0^h \times [0, \infty)$. Then for every natural number $n \in \mathbb{N}$, there is exactly one point

$$x(n) = (x_1(n), \ldots, x_k(n), y(n)) \in \mathcal{E}$$

satisfying $y(n) \in [n - 1, n)$. We may therefore assume without loss of generality that

$$\mathcal{E} = \{x(n) = (x_1(n), \ldots, x_k(n), n - 1) : n \in \mathbb{N}\}. \quad (62)$$

Proof of Theorem 6A. Let $p$ be a prime satisfying $p < k$. It follows from Lemma 1 that to prove Theorem 6A, it suffices to show that there are no 1-sets of class 1 with respect to the prime $p$ in $U_0^k \times [0, \infty)$. Suppose on the contrary that $\mathcal{E}$ is a 1-set of class 1 with respect to the prime $p$ in
where $b_i(n)$ are integers satisfying

$$0 \leq b_i(n) < p,$$

and where $0 \leq c_i(n) < p^{-1}$. Since $\mathcal{A}$ is a 1-set of class 1 with respect to the prime $p$ in $U_h^k \times [0, \infty)$, it follows that each elementary 1-box of order 1 with respect to the prime $p$ in $U_h^k \times [0, \infty)$ of the form

$$[0, 1)^{k-i} \times I \times [0, 1)^{i-1} \times [0, p),$$

where $I$ is an elementary $p$-type interval of order 1, contains exactly one point of $\mathcal{A}$, or, more precisely, exactly one point of

$$\{x(n): x(n) \in \mathcal{A} \land 1 \leq n \leq p\}.$$

It follows from (63) that for each $i = 1, \ldots, k$, the integers $b_i(1), \ldots, b_i(p)$ are distinct and form a complete set of residues modulo $p$. Hence by Lemma 9, we can assume, without loss of generality, that for each $i = 1, \ldots, k$ and each $n = 1, \ldots, p$,

$$b_i(n) = n - 1.$$

Consider now the $k$ integers $b_i(p+1), \ldots, b_k(p+1)$. Since $p < k$, we see that by (63) and Dirichlet’s box principle, there exist $j$ and $J$ satisfying $1 \leq j < J \leq k$ such that

$$b_j(p+1) = b_J(p+1),$$

and then follows from (64) and (65) that the box

$$[0, 1)^{k-j} \times [(b_j+1)p^{-1}] \times [0, 1)^{j-1} \times [b_j^{-1}, (b_j+1)p^{-1}) \times [0, 1)^{j-1} \times [0, p^2),$$

an elementary 1-box of order 1 with respect to the prime $p$ in $U_h^k \times [0, \infty)$, contains at least 2 points of $\mathcal{A}$, a contradiction. This completes the proof of Theorem 6A.

To prove Theorem 6B, the following idea is useful. Suppose $\mathcal{A}$ is a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_h^k \times [0, \infty)$, and that $\mathcal{A}$ is of the form (62).

**Definition.** We say that two natural numbers $n_1$ and $n_2$ are compatible with respect to the primes $p_1, \ldots, p_k$ if for each $j = 1, \ldots, k$,

$$[p_j^{-1}(n_1 - 1)] = [p_j^{-1}(n_2 - 1)],$$

where $[x]$ denotes the integer part of $x$. 

$U_h^k \times [0, \infty)$, and that $\mathcal{A}$ is of the form (62). For each $n \in \mathbb{N}$ and $i = 1, \ldots, k$, write

$$x_i(n) = b_i(n)p^{-1} + c_i(n),$$

where $b_i(n)$ are integers satisfying

$$0 \leq b_i(n) < p,$$
Lemma 10. Suppose $\mathcal{Q}$, of the form (62), is a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_0^* \times [0, \infty)$, and suppose $n_1$ and $n_2$ are compatible with respect to the same primes. Suppose $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by replacing the two points $x(n_1)$ and $x(n_2)$ by $(x_1(n_1), \ldots, x_k(n_1), n_2 - 1)$ and $(x_1(n_2), \ldots, x_k(n_2), n_1 - 1)$. Then $\mathcal{Q}'$ is also a 1-set of class $h$ with respect to the primes $p_1, \ldots, p_k$ in $U_0^* \times [0, \infty)$, and of the form (62).

Proof. Any elementary 1-box of order $h$ is of the form

$$I_1 \times \cdots \times I_k \times [mp_1^{s_1} \cdots p_k^{s_k}, (m + 1)p_1^{s_1} \cdots p_k^{s_k}],$$

where $m$ is a non-negative integer and where, for each $j = 1, \ldots, k$, $I_j$ is an elementary $p_j$-type interval of order $s_j$ satisfying $0 \leq s_j \leq h$. We distinguish two cases.

Case I. Suppose $s_1 = \cdots = s_k = 0$. Then $I_1 = \cdots = I_k = U_0$. It is easily seen that every box of the type $U_0^* \times [m, m + 1)$ contains exactly one point of $\mathcal{Q}'$.

Case II. Suppose $s_j > 0$ for some $j$ satisfying $1 \leq j \leq k$. Then if

$$x(n_j) \in I_1 \times \cdots \times I_k \times [mp_1^{s_1} \cdots p_k^{s_k}, (m + 1)p_1^{s_1} \cdots p_k^{s_k}],$$

we have that

$$n_j - 1 \in [\alpha p_1, (\alpha + 1)p_1) \subset [mp_1^{s_1} \cdots p_k^{s_k}, (m + 1)p_1^{s_1} \cdots p_k^{s_k}]$$

for some integer $\alpha$ satisfying

$$\alpha p_1 \leq x(n_j) \leq \alpha + (m + 1)p_1.$$

It follows from (66) that $n_j - 1 \in [\alpha p_1, (\alpha + 1)p_1)$, so that

$$(x_1(n_1), \ldots, x_k(n_1), n_2 - 1) \in I_1 \times \cdots \times I_k \times [mp_1^{s_1} \cdots p_k^{s_k}, (m + 1)p_1^{s_1} \cdots p_k^{s_k}].$$

The proof of Lemma 10 is now complete.

The proof of Theorem 6B is more complicated than that of Theorem 6A. This is because 1-sets of class 1 with respect to the primes 2, 2, 3 in $U_0^* \times [0, \infty)$ exist, so that we have to show that they cannot be generalized to form 1-sets of class 2 with respect to the same primes in $U_0^* \times [0, \infty)$.

Proof of Theorem 6B. We first make use of Lemmas 9 and 10 to reduce the problem to investigating special cases. Suppose $\mathcal{Q}$, of the form (62), is a 1-set of class 2 with respect to the primes 2, 2, 3 in $U_0^* \times [0, \infty)$, so that

$$\mathcal{Q} = \{x_1(n), x_2(n), x_3(n), n - 1) \in \mathbb{N} \}.$$

For each $n \in \mathbb{N}$ and for $i = 1, 2$, write

$$x_i(n) = b_{i,1}(n)2^{-1} + b_{i,2}(n)2^{-2} + c_i(n),$$
where \( b_{1,1}(n) \) and \( b_{1,2}(n) \) are integers 0 or 1, and \( 0 \leq c_4(n) < 2^{-2} \). Also, for each \( n \in \mathbb{N} \), write

\[
x_3(n) = b_3(n)3^{-1} + c_5(n),
\]

where \( b_3(n) \) are integers 0, 1 or 2 and \( 0 \leq c_5(n) < 3^{-1} \). In what follows, \( I_2 \) and \( I_3 \) will denote respectively arbitrary elementary 2- and 3-type intervals of order 1. We shall prove Theorem 6B in a number of steps.

(A) By Lemma 9, we may assume, without loss of generality, that \( b_{1,1}(4) = 1 \) and \( b_{1,2}(4) = 0 \). Then \( b_{1,1}(3) = 0 \), for otherwise the box \( \left[ \frac{1}{2}, 1 \right) \times U_0 \times U_0 \times [2, 4) \) contains 2 points. Similarly \( b_{2,1}(3) = 1 \).

(B) In order that each of the 4 boxes \( I_2 \times I_2 \times U_0 \times [0, 4) \) contains exactly one point, and since 1 and 2 are compatible with respect to the primes 2, 3, we can, in view of Lemma 10, assume that \( b_{1,1}(1) = b_{2,1}(1) = 0 \) and \( b_{1,1}(2) = b_{2,1}(2) = 1 \).

(C) In order that each of the 3 boxes \( U_0 \times U_0 \times I_3 \times [0, 3) \) contains exactly one point, \( b_3(1) \), \( b_3(2) \) and \( b_3(3) \) must be distinct. By Lemma 9, we may assume, without loss of generality, that \( b_3(n) = n - 1 \) for \( n = 1, 2, 3 \).

(D) Suppose \((b_{1,1}(5), b_{2,1}(5)) = (0, 1)\). Then \( b_3(5) = 1 \), for otherwise the box \( [0, \frac{1}{2}) \times U_0 \times [\frac{1}{2}, 1) \times [0, 6) \) contains no points. But \( b_3(5) \neq 1 \), for otherwise the box \( U_0 \times [\frac{1}{2}, 1) \times [\frac{1}{2}, 1) \times [0, 6) \) contains 2 points. Since 5 and 6 are compatible with respect to the primes 2, 3, \((b_{1,1}(5), b_{2,1}(5)) \neq (1, 0)\) either. We may assume, therefore, that \( b_{1,1}(5) = b_{2,1}(5) = 0 \) and \( b_{1,1}(6) = b_{2,1}(6) = 1 \).

(E) It follows that \( b_3(6) = 1 \), \( b_3(4) = 2 \) and so \( b_3(6) = 0 \).

(F) As in (B), we may assume, without loss of generality, that \( b_{1,1}(7) = b_{1,2}(8) = 0 \) and \( b_{1,2}(8) = b_3(7) = 1 \).

(G) In order that each of the 4 boxes \( I_2 \times I_2 \times U_0 \times [8, 12) \) contains exactly one point, there exist \( n_1 \) and \( n_2 \) satisfying \( 9 \leq n_1 \leq 12, 9 \leq n_2 \leq 12 \) and \( n_1 \neq n_2 \) such that \( b_{1,1}(n_1) = b_{2,1}(n_1) = 0, b_{1,1}(n_2) = b_{2,1}(n_2) = 1 \). Then \( b_3(n_1) = 2 \), for otherwise the box \( [0, \frac{1}{2}) \times [0, \frac{1}{2}) \times [0, 12] \) contains no points. Similarly, \( b_3(n_2) = 2 \). It follows that either \( n_1 = 9 \) or \( n_2 = 9 \), for otherwise the box \( U_0 \times U_0 \times [\frac{1}{2}, 1) \times [9, 12) \) contains at least 2 points. Hence \( b_3(9) = 2 \).

(H) By Lemma 9, we may assume, without loss of generality, that \( b_{1,2}(1) = 0 \). Then \( b_{1,2}(3) = 1 \), for otherwise the box \( [0, \frac{1}{2}) \times U_0 \times U_0 \times [0, 4) \) contains 2 points. Also, \( b_3(5) = 1 \), for otherwise the box \( [0, \frac{1}{2}) \times [0, \frac{1}{2}) \times U_0 \times [0, 8) \) contains 2 points. It follows that \( b_{1,2}(7) = 0 \).

(I) It follows that \( b_3(7) \neq 0 \), for otherwise the box \( [0, \frac{1}{2}) \times U_0 \times [0, \frac{1}{2}) \times [0, 12) \) contains 2 points. Since \( b_3(9) = 2 \), we cannot have \( b_3(7) = 2 \). It follows that \( b_3(7) = 1 \). Hence \( b_3(8) = 0 \).
(J) By Lemma 9, we may assume, without loss of generality, that $b_{2,2}(1) = 0$. Then $b_{2,2}(5) = 1$, for otherwise the box $[0, \frac{1}{2}] \times [0, \frac{1}{2}] \times U_0 \times [0, 8)$ contains 2 points.

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(K) We cannot have $b_{2,2}(8) = 1$, for otherwise the box $U_0 \times [\frac{1}{2}, \frac{1}{2}] \times U_0 \times [4, 8)$ contains 2 points. On the other hand, we cannot have $b_{2,2}(8) = 0$, for otherwise the box $U_0 \times [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [0, 12)$ contains 2 points.

This completes the proof of Theorem 6B.

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