Chapter 10

PARAMETRIZED SURFACES

10.1. Introduction

In this chapter, we discuss parametrized surfaces. Recall that a path is essentially a parametrization of a curve. In a similar way, a surface can be parametrized. Whereas a curve can be parametrized by the use of a single real parameter, a surface can be parametrized by the use of two real parameters. We shall be concerned only with surfaces in $\mathbb{R}^3$. Before we give any formal definition, let us consider two examples.

Example 10.1.1. Consider a function $f : [A, B] \times [C, D] \to \mathbb{R}$. Then the graph
\[
\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D] \text{ and } z = f(x, y)\}
\]
is a surface in $\mathbb{R}^3$. Each point $(x, y, z)$ on this surface is determined precisely by the values of the variables $x$ and $y$.

Example 10.1.2. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, with radius 1 and centre $(0, 0, 0)$. Using spherical coordinates, we can write
\[
x = \sin \phi \cos \theta, \quad y = \sin \phi \sin \theta, \quad z = \cos \phi,
\]
where $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Each point $(x, y, z)$ on the sphere is determined precisely by the values of the variables $\phi$ and $\theta$.

Definition. By a parametrized surface in $\mathbb{R}^3$, we mean a function of the type
\[
\Phi : R \to \mathbb{R}^3,
\]
where $R \subseteq \mathbb{R}^2$ is a domain. The range
\[
\Phi(R) = \{\Phi(u, v) : (u, v) \in R\} \subseteq \mathbb{R}^3
\]
of the function $\Phi$ is called a surface.
Remarks. (1) For every \((u, v) \in \mathbb{R}\), we can write \(\Phi(u, v) = (x(u, v), y(u, v), z(u, v))\), with components \(x(u, v), y(u, v), z(u, v) \in \mathbb{R}\).

(2) We can think of the function \(\Phi\) as twisting and bending the region \(R\) to give a surface \(S = \Phi(R)\). The position of a point \((x(u, v), y(u, v), z(u, v))\) on \(S\) is determined by the values of the parameters \(u\) and \(v\).

(3) Often, we refer to the parametrized surface \(\Phi(u, v)\) without specifying the domain of definition of the function \(\Phi\). This is a convenient abuse of rigour.

Definition. We say that a parametrized surface \(\Phi : \mathbb{R} \to \mathbb{R}^3\) is continuously differentiable if the function \(\Phi\) is differentiable and the partial derivatives are continuous.

Suppose that a parametrized surface \(\Phi : \mathbb{R} \to \mathbb{R}^3\) is differentiable at a point \((u_0, v_0) \in \mathbb{R}\). Keeping the first parameter \(u\) fixed at \(u_0\), we consider the function \(v \mapsto \Phi(u_0, v)\) in a neighbourhood of \(v_0\). The image of this function is a curve on the surface \(S = \Phi(R)\), and a tangent vector to this curve at the point \(\Phi(u_0, v_0)\) is given by

\[
t_u(u_0, v_0) = \left( \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right).
\]

On the other hand, keeping the second parameter \(v\) fixed at \(v_0\), we consider the function \(u \mapsto \Phi(u, v_0)\) in a neighbourhood of \(u_0\). The image of this function is a curve on the surface \(S = \Phi(R)\), and a tangent vector to this curve at the point \(\Phi(u_0, v_0)\) is given by

\[
t_v(u_0, v_0) = \left( \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right).
\]

It follows that \(t_u(u_0, v_0)\) and \(t_v(u_0, v_0)\) are two vectors tangent to the surface \(S = \Phi(R)\) at the point \(\Phi(u_0, v_0)\). Unless they are parallel or opposite, these two vectors determine the tangent plane to the surface \(S = \Phi(R)\) at the point \(\Phi(u_0, v_0)\). In this case, the vector \(t_u(u_0, v_0) \times t_v(u_0, v_0)\) is a vector normal to the surface \(S = \Phi(R)\) at the point \(\Phi(u_0, v_0)\).

Definition. We say that a parametrized surface \(\Phi : \mathbb{R} \to \mathbb{R}^3\) is smooth at a point \(\Phi(u_0, v_0)\) if

\[
t_u(u_0, v_0) \times t_v(u_0, v_0) \neq 0.
\]

We say that a parametrized surface \(\Phi : \mathbb{R} \to \mathbb{R}^3\) is smooth if it is smooth at every point \(\Phi(u_0, v_0)\) where \((u_0, v_0) \in \mathbb{R}\); in other words, if \(t_u \times t_v \neq 0\) in \(\mathbb{R}\).
Example 10.1.3. For the parametrized cone

\[ \Phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, u), \]

we have

\[ t_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (\cos v, \sin v, 1) \quad \text{and} \quad t_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (-u \sin v, u \cos v, 0). \]

Hence

\[ t_u \times t_v = (\cos v, \sin v, 1) \times (-u \sin v, u \cos v, 0) = (-u \cos v, -u \sin v, u). \]

It follows that the parametrized cone is smooth everywhere except at \((0, 0, 0)\).

Example 10.1.4. For the parametrized sphere

\[ \Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u), \]

we have

\[ t_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (\cos u \cos v, \cos u \sin v, -\sin u) \]

and

\[ t_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (-\sin u \sin v, \sin u \cos v, 0). \]

Hence

\[ t_u \times t_v = (\cos u \cos v, \cos u \sin v, -\sin u) \times (-\sin u \sin v, \sin u \cos v, 0) \]

\[ = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u). \]

It follows that the parametrized sphere is smooth everywhere except at \((0, 0, \pm 1)\). A similar argument shows that the parametrized sphere \(\Psi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3 : (u, v) \mapsto (\cos u, \sin u \cos v, \sin u \sin v)\) is smooth everywhere except at \((\pm 1, 0, 0)\). Note that both \(\Phi\) and \(\Psi\) are parametrizations of the same unit sphere \(x^2 + y^2 + z^2 = 1\).

Example 10.1.5. For the parametrized surface \(\Phi : [-1, 1] \times [-1, 1] \to \mathbb{R}^3 : (u, v) \mapsto (u, v, 0)\), we have

\[ t_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (1, 0, 0) \quad \text{and} \quad t_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (0, 1, 0). \]

Hence

\[ t_u \times t_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1). \]

It follows that the parametrized surface is smooth everywhere. Note that \(\Phi([-1, 1] \times [-1, 1])\) is the square with vertices \((\pm 1, \pm 1, 0)\). For the parametrized surface \(\Psi : [-1, 1] \times [-1, 1] \to \mathbb{R}^3 : (u, v) \mapsto (u^3, v^3, 0)\), we have

\[ t_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (3u^2, 0, 0) \quad \text{and} \quad t_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (0, 3v^2, 0). \]

Hence

\[ t_u \times t_v = (3u^2, 0, 0) \times (0, 3v^2, 0) = (0, 0, 9u^2v^2). \]
It follows that the parametrized surface is smooth everywhere except at points \( \Psi(u, v) \) where \( u = 0 \) or \( v = 0 \). Note that \( \Psi([-1, 1] \times [-1, 1]) \) is also the square with vertices \((\pm 1, \pm 1, 0)\).

REMARKS. (1) Examples 10.1.4 and 10.1.5 show that smoothness depends on the parametrization and not just the surface. Indeed, we say that a surface \( S \) is smooth if there exists a parametrization \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) which is smooth and such that \( \Phi(I) = S \).

(2) Suppose that a parametrized surface \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) is smooth at a point \((x_0, y_0, z_0) = \Phi(u_0, v_0)\). Then \( \mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v \), evaluated at \((u_0, v_0)\), is a normal vector to the surface \( S = \Phi(I) \) at \((x_0, y_0, z_0)\). It follows that the equation of the tangent plane to \( S \) at \((x_0, y_0, z_0)\) is given by \((x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} = 0\), where \( \mathbf{n} \) is evaluated at \((u_0, v_0)\).

Example 10.1.6. Suppose that \( f : [A, B] \times [C, D] \to \mathbb{R} \) is a differentiable function. Then its graph
\[
\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in [A, B] \times [C, D] \text{ and } z = f(x, y)\}
\]
is the range of the function \( \Phi : [A, B] \times [C, D] \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v)) \). We have
\[
\mathbf{t}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = \left( 1, 0, \frac{\partial f}{\partial u} \right) \quad \text{and} \quad \mathbf{t}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = \left( 0, 1, \frac{\partial f}{\partial v} \right).
\]
Hence
\[
\mathbf{t}_u \times \mathbf{t}_v = \begin{pmatrix} 1 & 0 & \frac{\partial f}{\partial u} \\ 0 & 1 & \frac{\partial f}{\partial v} \end{pmatrix} \begin{pmatrix} 0 & -\frac{\partial f}{\partial u} \\ 1 & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} \end{pmatrix} \neq 0.
\]
It follows that the surface is smooth.

Example 10.1.7. The hyperboloid of one sheet is given by the equation \( x^2 + y^2 - z^2 = 1 \). We can write
\[
x = r \cos u \quad \text{and} \quad y = r \sin u,
\]
so that \( r^2 - z^2 = 1 \). We can then write
\[
r = \cosh v \quad \text{and} \quad z = \sinh v.
\]
Hence we have the parametrization
\[
x = \cos u \cosh v, \quad y = \sin u \cosh v, \quad z = \sinh v.
\]
Consider now the function \( \Phi(u, v) = (\cos u \cosh v, \sin u \cosh v, \sinh v) \). We have
\[
\mathbf{t}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) = (-\sin u \cosh v, \cos u \cosh v, 0)
\]
and
\[
\mathbf{t}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (\cos u \sinh v, \sin u \sinh v, \cosh v).
\]
Hence
\[
\mathbf{t}_u \times \mathbf{t}_v = (-\sin u \cosh v, \cos u \cosh v, 0) \times (\cos u \sinh v, \sin u \sinh v, \cosh v)
\]
\[
= (\cos u \cosh^2 v, \sin u \cosh^2 v, -\cosh v \sinh v) \neq 0.
\]
It follows that this surface is smooth.
10.2. Surface Area

For the remainder of these notes, we restrict our attention to piecewise smooth surfaces. These are finite unions of the ranges of parametrized surfaces of the type \( \Phi_i : \mathbb{R}^i \to \mathbb{R}^3 \), where \( \mathbb{R}^i \) is an elementary region in \( \mathbb{R}^2 \), \( \Phi_i \) is continuously differentiable and one-to-one, except possibly on the boundary of \( \mathbb{R}^i \), and \( S_i = \Phi(\mathbb{R}^i) \) is smooth, except possibly at a finite number of points.

For a discussion of elementary regions in \( \mathbb{R}^2 \), the reader is referred to Section 5.4.

**Definition.** Suppose that \( \Phi : \mathbb{R}^2 \to \mathbb{R}^3 \) is a continuously differentiable parametrized surface. Then the quantity

\[
A = \int \int \limits_{\mathbb{R}} \| \mathbf{t}_u \times \mathbf{t}_v \| \, du \, dv
\]

is called the surface area of the parametrized surface \( \Phi \).

**Remarks.** (1) Note that

\[
\mathbf{t}_u \times \mathbf{t}_v = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \times \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)
\]

\[
= \left( \frac{\partial y \partial z - \partial z \partial y}{\partial u \partial v}, \frac{\partial z \partial x - \partial x \partial z}{\partial u \partial v}, \frac{\partial x \partial y - \partial y \partial x}{\partial u \partial v} \right)
\]

and so

\[
\| \mathbf{t}_u \times \mathbf{t}_v \| = \left( \left| \frac{\partial (y, z)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (z, x)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (x, y)}{\partial (u, v)} \right|^2 \right)^{1/2} = \left( \left| \frac{\partial (x, y)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (x, z)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (y, z)}{\partial (u, v)} \right|^2 \right)^{1/2}.
\]

Hence

\[
A = \int \int \limits_{\mathbb{R}} \left( \left| \frac{\partial (x, y)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (x, z)}{\partial (u, v)} \right|^2 + \left| \frac{\partial (y, z)}{\partial (u, v)} \right|^2 \right)^{1/2} \, du \, dv.
\]

(2) To justify the definition, consider a small rectangle in \( \mathbb{R}^2 \) with bottom left vertex \((u, v)\) and top right vertex \((u + \Delta u, v + \Delta v)\).

The image under \( \Phi \) of this rectangle can be approximated by a parallelogram in \( \mathbb{R}^3 \), with area

\[
\| \Delta u \mathbf{t}_u \times \Delta v \mathbf{t}_v \| = \| \mathbf{t}_u \times \mathbf{t}_v \| \, \Delta u \Delta v.
\]
Example 10.2.1. For the parametrized cone
\[
\Phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, u),
\]
we have
\[
t_u \times t_v = (-u \cos v, -u \sin v, u),
\]
so that
\[
\|t_u \times t_v\| = (u^2 \cos^2 v + u^2 \sin^2 v + u^2)^{1/2} = \sqrt{2}u.
\]
Alternatively, we have
\[
\|t_u \times t_v\| = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{1/2} = (u^2 + u^2 \sin^2 v + u^2 \cos^2 v)^{1/2} = \sqrt{2}u.
\]
Hence the surface area is given by
\[
A = \int_0^{2\pi} \left(\int_0^1 \sqrt{2}u \, du\right) \, dv = \pi \sqrt{2}.
\]

Example 10.2.2. For the parametrized sphere
\[
\Phi : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (\sin u \cos v, \sin u \sin v, \cos u),
\]
we have
\[
t_u \times t_v = (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u),
\]
so that
\[
\|t_u \times t_v\| = (\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u)^{1/2} = |\sin u|.
\]
Alternatively, we have
\[
\|t_u \times t_v\| = \left(\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2\right)^{1/2} = (\cos^2 u \sin^2 v + \sin^4 u \sin^2 v + \sin^4 u \cos^2 v)^{1/2} = |\sin u|.
\]
Hence the surface area is given by
\[
A = \int_0^{2\pi} \left(\int_0^\pi |\sin u| \, du\right) \, dv = 2\int_0^{2\pi} \left(\int_0^\pi/2 \sin u \, du\right) \, dv = 4\pi.
\]

Example 10.2.3. For the helicoid
\[
\Phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, v),
\]
we have
\[
t_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 0) \quad \text{and} \quad t_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u \sin v, u \cos v, 1).
Hence
\[ \mathbf{t}_u \times \mathbf{t}_v = (\cos v, \sin v, 0) \times (-u \sin v, u \cos v, 1) = (\sin v, -\cos v, u), \]
so that
\[ \|\mathbf{t}_u \times \mathbf{t}_v\| = (\sin^2 v + \cos^2 v + u^2)^{1/2} = (1 + u^2)^{1/2}. \]

Alternatively, we have
\[ \|\mathbf{t}_u \times \mathbf{t}_v\| = \left( \left| \frac{\partial(x, y)}{\partial(u, v)} \right|^2 + \left| \frac{\partial(x, z)}{\partial(u, v)} \right|^2 + \left| \frac{\partial(y, z)}{\partial(u, v)} \right|^2 \right)^{1/2} = (u^2 + \cos^2 v + \sin^2 v)^{1/2} = (u^2 + 1)^{1/2}. \]

Hence the surface area is given by
\[ A = \int^0_{2\pi} \left( \int^1_0 \left(1 + u^2\right)^{1/2} du \right) dv = \pi (\sqrt{2} + \log(1 + \sqrt{2})). \]

**Example 10.2.4.** Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a continuously differentiable function, where \( R \subseteq \mathbb{R}^2 \) is an elementary region. Then its graph
\[ \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R \text{ and } z = f(x, y)\} \]
is the range of the function \( \Phi : \mathbb{R} \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v)) \), and
\[ \mathbf{t}_u \times \mathbf{t}_v = \left( -\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1 \right). \]

Hence the surface area of the graph is
\[ A = \iint_R \left( \left( \frac{\partial f}{\partial u} \right)^2 + \left( \frac{\partial f}{\partial v} \right)^2 + 1 \right)^{1/2} dudv. \]
Chapter 10: Parametrized Surfaces

Problems for Chapter 10

1. For each of the following parametrized surfaces, find a unit vector normal to the surface at a point \( \Phi(u, v) \):
   a) \( \Phi : [0, \pi] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (3 \sin u \cos v, 2 \sin u \sin v, \cos u) \)
   b) \( \Phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (\sin v, u, \cos v) \)
   c) \( \Phi : [-\pi, \pi] \times [-\pi, \pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto ((2 - \cos u) \sin v, (2 - \cos u) \cos v, \sin u) \)
   d) \( \Phi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u, v, v) \)

2. For each of the following parametrized surfaces, find the equation of the tangent plane, if it exists, at the point given, and determine also whether the surface is smooth:
   a) \( \Phi : [-1, 1] \times [0, 2] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u^2 + v, v^2, 2u) \); at \( \Phi(0, 1) \)
   b) \( \Phi : [-1, 1] \times [0, 2] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u^2 - v^2, u + v, u^2 + 4v) \); at \( \Phi(0, 1) \)
   c) \( \Phi : [0, 2] \times [-\pi, \pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u^2 \cos v, u^2 \sin v, u) \); at \( \Phi(1, 0) \)

3. Consider the hyperboloid \( x^2 - y^2 + z^2 = 9 \).
   a) Find a parametrization of the hyperboloid.
   b) Use your parametrization in part (a) to find a unit normal to the surface.
   c) Find the equation of the tangent plane to the surface at a point \( \Phi(x_0, y_0, z_0) \), where \( x_0^2 + y_0^2 = 9 \).
   d) Show that the lines \( (x_0, y_0, z_0) + \lambda(-2z_0, 3, x_0) \) and \( (x_0, y_0, z_0) + \lambda(-3z_0, 3, -x_0) \) are part of the surface as well as the tangent plane in part (c).

4. Consider a parametrized surface \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), smooth at a point \( \Phi(u_0, v_0) \).
   a) Show that the linear approximation
      \[
      (u, v) \mapsto \Phi(u_0, v_0) + (D\Phi)(u_0, v_0) \left( \begin{array}{c} u - u_0 \\ v - v_0 \end{array} \right)
      \]
      represents a plane through the point \( \Phi(u_0, v_0) \).
      b) Show that the plane in part (a) is the tangent plane to the surface at \( \Phi(u_0, v_0) \).

5. Consider the paraboloid parametrized by \( \Phi : [0, 2] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, u^2) \).
   a) Find an equation in \( x, y, z \) describing the surface.
   b) Find a unit vector orthogonal to the surface at \( \Phi(u, v) \).
   c) Find the surface area.

6. Let \( R \subseteq \mathbb{R}^2 \) be the unit disc with centre \((0, 0)\) and radius 1. Find the area of the parametrized surface \( \Phi : R \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u - v, u + v, uv) \).

7. Find the area of the parametrized surface \( \Phi : [0, 1] \times [0, 2\pi] : (u, v) \mapsto (u \cos v, 2u \cos v, u) \).

8. Find a parametrization of the ellipsoid
      \[
      \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,
      \]
      and express the surface area of the (parametrized) ellipsoid as an integral.

9. Suppose that \( a > 0 \) is fixed. By using a suitable parametrization, find the surface area of the part of the cylinder \( x^2 + z^2 = a^2 \) that is inside the cylinder \( x^2 + y^2 = 2ay \) and also in the first octant \( x, y, z \geq 0 \).

10. Suppose that \( g : [A, B] \rightarrow \mathbb{R} \) is a continuous function such that \( g(x) \geq 0 \) for every \( x \in [A, B] \). Now rotate the graph \( \{(x, g(x)) : x \in [A, B]\} \) about the x-axis. Follow the steps below to calculate the surface area generated by this rotation.
    a) Convince yourself that the function \( \Phi : [A, B] \times [0, 2\pi] \rightarrow \mathbb{R}^3 : (u, v) \mapsto (u, g(u) \cos v, g(u) \sin v) \)
       is a parametrization of the surface of revolution.
    b) Show that the surface area of \( \Phi \) is given by
       \[
       A = 2\pi \int_A^B g(u)(1 + |g'(u)|^2)^{1/2} \, du.
       \]