Chapter 9

INTEGRALS OVER PATHS

9.1. Integrals of Scalar Functions over Paths

Suppose that the path

\( \phi : [A, B] \rightarrow \mathbb{R}^n : t \mapsto \phi(t) = (x_1(t), \ldots, x_n(t)) \)

is continuously differentiable. For any real valued function \( f(x_1, \ldots, x_n) \) such that the composition function

\( f \circ \phi : [A, B] \rightarrow \mathbb{R} : t \mapsto f(x_1(t), \ldots, x_n(t)) \)

is continuous, we define

\[
\int_{\phi} f \, ds = \int_{\phi} f(x_1, \ldots, x_n) \, ds = \int_{A}^{B} f(\phi(t)) \| \phi'(t) \| \, dt.
\]

Remarks. (1) We are mainly interested in the special cases \( n = 2 \) and \( n = 3 \), and write respectively

\[
\int_{\phi} f \, ds = \int_{\phi} f(x, y) \, ds \quad \text{and} \quad \int_{\phi} f \, ds = \int_{\phi} f(x, y, z) \, ds.
\]

(2) Suppose that \( f = 1 \) identically. Then the integral simply represents the arc length of \( \phi \).

(3) Note that \( f \) has only to be defined on the image curve \( C = \phi([A, B]) \) of the path \( \phi \) for our definition to make sense. The continuity of the composition function \( f \circ \phi \) on the closed interval \([A, B]\) ensures the existence of the integral.
(4) Sometimes, $\phi$ may only be piecewise continuously differentiable; in other words, there exists a dissection $A = t_0 < t_1 < \ldots < t_k = B$ of the interval $[A, B]$ such that $\phi$ is continuously differentiable in $[t_{i-1}, t_i]$ for each $i = 1, \ldots, k$. In this case, we define

$$\int_{\phi} f \, ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\phi(t)) \|\phi'(t)\| \, dt.$$  

In other words, we calculate the corresponding integral for each subinterval and consider the sum of the integrals.

(5) For the special case $n = 2$, we must not confuse the integral with integrals of the type

$$\int_{\phi} f(z) \, dz$$

which arise frequently in complex analysis.

**Example 9.1.1.** Suppose that $\phi : [0, 2\pi] \rightarrow \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$ and $f(x, y, z) = x + y + z$. Then

$$\int_{\phi} f \, ds = \int_{0}^{2\pi} f(\cos t, \sin t, t) \sqrt{1 + \cos^2 t + \sin^2 t} \, dt = \int_{0}^{2\pi} (\cos t + \sin t + t) \, dt = 2\pi^2 \sqrt{2}.$$  

**Example 9.1.2.** Suppose that $\phi : [0, 2\pi] \rightarrow \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)$ and $f(x, y) = \sqrt{2}y$. Then

$$\int_{\phi} f \, ds = \int_{0}^{2\pi} f(t - \sin t, 1 - \cos t) \sqrt{1 + \sin^2 t + \cos^2 t} \, dt = \int_{0}^{2\pi} (2 - 2 \cos t) \, dt = 4\pi.$$  

**Example 9.1.3.** Suppose that $\phi : [0, \pi] \rightarrow \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t)$ and $f(x, y) = 2 + 8y^2$. Then

$$\int_{\phi} f \, ds = \int_{0}^{\pi} f(\cos^3 t, \sin^3 t) \sqrt{3\cos^2 t(\cos^2 t + \sin^2 t)} \, dt = \int_{0}^{\pi} (2 + 8 \sin^6 t) \, dt = \int_{0}^{\pi} (6 + 24 \sin^6 t) \, dt = 12.$$  

**Example 9.1.4.** The three distinct paths

$$\phi : [0, 2\pi] \rightarrow \mathbb{R}^2 : t \mapsto (\cos t, -\sin t)$$

$$\psi : [0, 1] \rightarrow \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$$

$$\eta : [0, 1] \rightarrow \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2)$$

satisfy $\phi([0, 2\pi]) = \psi([0, 1]) = \eta([0, 1]) = C$, the unit circle in $\mathbb{R}^2$. Note also that the path $\phi$ follows $C$ in a clockwise direction, while the paths $\psi$ and $\eta$ follow $C$ in an anticlockwise direction. Now consider the function $f(x, y) = 1 + x + y$. Then

$$\int_{\phi} f \, ds = \int_{0}^{2\pi} f(\cos t, -\sin t) \, dt = \int_{0}^{2\pi} (1 + \cos t - \sin t) \, dt = 2\pi,$$

$$\int_{\psi} f \, ds = \int_{0}^{1} f(\cos 2\pi t, \sin 2\pi t) \, dt = \int_{0}^{1} (1 + \cos 2\pi t + \sin 2\pi t) \, dt = 2\pi,$$

$$\int_{\eta} f \, ds = \int_{0}^{1} f(\cos 2\pi t^2, \sin 2\pi t^2) \, dt = \int_{0}^{1} (1 + \cos 2\pi t^2 + \sin 2\pi t^2) \, dt = 2\pi.$$  

Note that all three integrals have the same value. We shall show in Section 9.3 that this is not just a coincidence.
9.2. Line Integrals

Suppose that the path
\[ \phi : [A, B] \to \mathbb{R}^n : t \mapsto \phi(t) = (x_1(t), \ldots, x_n(t)) \]
is continuously differentiable. For any vector field \( F(x_1, \ldots, x_n) \) such that the composition function
\[ F \circ \phi : [A, B] \to \mathbb{R}^n : t \mapsto F(x_1(t), \ldots, x_n(t)) \]
is continuous, we define
\[ \int_{\phi} F \cdot ds = \int_{\phi} F(x_1, \ldots, x_n) \cdot ds = \int_{A}^{B} F(\phi(t)) \cdot \phi'(t) \, dt. \]

**Remarks.**

(1) We are mainly interested in the special cases \( n = 2 \) and \( n = 3 \), and write respectively
\[ \int_{\phi} F \cdot ds = \int_{\phi} F(x, y) \cdot ds \quad \text{and} \quad \int_{\phi} F \cdot ds = \int_{\phi} F(x, y, z) \cdot ds. \]

Writing \( F = (F_1, F_2) \) and \( ds = (dx, dy) \) in the case \( n = 2 \) and \( F = (F_1, F_2, F_3) \) and \( ds = (dx, dy, dz) \) in the case \( n = 3 \), we have respectively
\[ \int_{\phi} F \cdot ds = \int_{\phi} (F_1, F_2) \cdot (dx, dy) = \int_{\phi} (F_1 \, dx + F_2 \, dy) = \int_{A}^{B} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} \right) \, dt \]
and
\[ \int_{\phi} F \cdot ds = \int_{\phi} (F_1, F_2, F_3) \cdot (dx, dy, dz) = \int_{\phi} (F_1 \, dx + F_2 \, dy + F_3 \, dz) = \int_{A}^{B} \left( F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) \, dt. \]

(2) Note that \( F \) has only to be defined on the image curve \( C = \phi([A, B]) \) of the path \( \phi \) for our definition to make sense. The continuity of the composition function \( F \circ \phi \) on the closed interval \([A, B]\) ensures the existence of the integral.

(3) Sometimes, \( \phi \) may only be piecewise continuously differentiable. As in the last section, we can calculate the corresponding integral for each subinterval in a dissection of the interval \([A, B]\) and consider the sum of the integrals.

(4) Note that if \( \phi'(t) \neq 0 \) for any \( t \in [A, B] \), then
\[ \int_{\phi} F \cdot ds = \int_{A}^{B} \left( F(\phi(t)) \cdot \frac{\phi'(t)}{\|\phi'(t)\|} \right) \|\phi'(t)\| \, dt = \int_{A}^{B} f(\phi(t)) \|\phi'(t)\| \, dt, \]
where
\[ f(\phi(t)) = F(\phi(t)) \cdot \frac{\phi'(t)}{\|\phi'(t)\|}. \]
Here \( \phi'(t)/\|\phi'(t)\| \) denotes the unit tangent vector along the path \( \phi \). The integral now becomes one of the type discussed in the last section.

(5) Suppose that \( F \) is a force field; for example, gravitational field or magnetic field. Suppose also that a particle is moving along a path \( \phi \). At any time \( t \), the force on the particle will be given by \( F(\phi(t)) \). On the other hand, a small displacement in the time interval \([t, t + dt]\) can be described by the velocity differential \( ds = \phi'(t) \, dt \). It follows that the scalar product \( F(\phi(t)) \cdot \phi'(t) \, dt \) denotes the work done in the time interval \([t, t + dt]\). Hence the integral describes the total work done.
Example 9.2.1. Suppose that \( \phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t) \) and \( F(x, y, z) = (x, y, z) \). Then

\[
\int_0^{2\pi} F \cdot ds = \int_0^{2\pi} F(\cos t, \sin t, t) \cdot (-\sin t, \cos t, 1) \, dt
= \int_0^{2\pi} (\cos t, \sin t, t) \cdot (-\sin t, \cos t, 1) \, dt
= \int_0^{2\pi} t \, dt = 2\pi^2.
\]

Example 9.2.2. Suppose that \( \phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t) \) and \( F(x, y) = (y, -x) \). Then

\[
\int_0^{2\pi} F \cdot ds = \int_0^{2\pi} F(t - \sin t, 1 - \cos t) \cdot (1 - \cos t, \sin t) \, dt
= \int_0^{2\pi} (1 - \cos t, \sin t - t) \cdot (1 - \cos t, \sin t) \, dt
= \int_0^{2\pi} (2 - 2 \cos t - t \sin t) \, dt = 6\pi.
\]

Example 9.2.3. Suppose that \( \phi : [0, \pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t) \) and \( F(x, y) = (-y, x) \). Then

\[
\int_0^\pi F \cdot ds = \int_0^\pi F(\cos^3 t, \sin^3 t) \cdot (-3 \cos^2 t \sin t, 3 \sin^2 t \cos t) \, dt
= \int_0^\pi (-\sin^3 t, \cos^3 t) \cdot (-3 \cos^2 t \sin t, 3 \sin^2 t \cos t) \, dt
= \int_0^\pi 3 \sin^2 t \cos^2 t \, dt = \frac{3}{4} \int_0^\pi \sin^2 2t \, dt = \frac{3}{8} \int_0^{2\pi} (1 - \cos 4t) \, dt = \frac{3\pi}{8}.
\]

Example 9.2.4. The three distinct paths

\[
\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t)
\]
\[
\psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t)
\]
\[
\eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (2 \cos 2\pi t, 2 \sin 2\pi t)
\]
satisfy \( \phi([0, 2\pi]) = \psi([0, 1]) = \eta([0, 1]) = C \), the unit circle in \( \mathbb{R}^2 \). Note also that the path \( \phi \) follows \( C \) in a clockwise direction, while the paths \( \psi \) and \( \eta \) follow \( C \) in an anticlockwise direction. Now consider the vector field \( F(x, y) = (-y, x) \). Then

\[
\int_\phi F \cdot ds = \int_0^{2\pi} F(\cos t, -\sin t) \cdot (-\sin t, -\cos t) \, dt
= \int_0^{2\pi} (\cos t, \sin t) \cdot (-\sin t, -\cos t) \, dt = \int_0^{2\pi} (-1) \, dt = -2\pi,
\]

\[
\int_\psi F \cdot ds = \int_0^1 F(\cos 2\pi t, \sin 2\pi t) \cdot (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t) \, dt
= \int_0^1 (-\sin 2\pi t, \cos 2\pi t) \cdot (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t) \, dt = \int_0^1 2\pi \, dt = 2\pi,
\]

\[
\int_\eta F \cdot ds = \int_0^1 F(2 \cos 2\pi t, 2 \sin 2\pi t) \cdot (-4\pi t \sin 2\pi t^2, 4\pi t \cos 2\pi t^2) \, dt
= \int_0^1 (-\sin 2\pi t^2, \cos 2\pi t^2) \cdot (-4\pi t \sin 2\pi t^2, 4\pi t \cos 2\pi t^2) \, dt = \int_0^1 4\pi t \, dt = 2\pi.
\]
Note that
\[-\int_{\phi} F \cdot ds = \int_{\psi} F \cdot ds = \int_{\eta} F \cdot ds,
\]
where \(\psi\) and \(\eta\) follow the unit circle \(C\) in the same direction while \(\phi\) follows the unit circle \(C\) in the opposite direction. We shall show in Section 9.3 that this is not just a coincidence.

**Example 9.2.5.** The three distinct paths

\[
\phi : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, t) \\
\psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, t^2) \\
\eta : [0, \pi/2] \to \mathbb{R}^2 : t \mapsto (1 - \cos t, \sin t)
\]

all have the same initial point \((0, 0)\) and the same terminal point \((1, 1)\). The curve \(\phi([0, 1])\) is part of the straight line \(y = x\), the curve \(\psi([0, 1])\) is part of the parabola \(y = x^2\), while the curve \(\eta([0, \pi/2])\) is part of the circle \((x - 1)^2 + y^2 = 1\). Hence the three paths have different curves. Consider now the vector field \(F(x, y) = (y, x)\). Then

\[
\int_{\phi} F \cdot ds = \int_{0}^{1} (t, t) \cdot (1, 1) \, dt = \int_{0}^{1} (t, t) \cdot (1, 1) \, dt = \int_{0}^{1} 2t \, dt = 1,
\]
\[
\int_{\psi} F \cdot ds = \int_{0}^{1} (t, t^2) \cdot (1, 2t) \, dt = \int_{0}^{1} (t^2, t) \cdot (1, 2t) \, dt = \int_{0}^{1} 3t^2 \, dt = 1,
\]
\[
\int_{\eta} F \cdot ds = \int_{0}^{\pi/2} (1 - \cos t, \sin t) \cdot (\sin t, \cos t) \, dt = \int_{0}^{\pi/2} (\sin t, 1 - \cos t) \cdot (\sin t, \cos t) \, dt
\]
\[
= \int_{0}^{\pi/2} (\cos t + \sin^2 t - \cos^2 t) \, dt = \int_{0}^{\pi/2} (\cos t - \cos 2t) \, dt = 1.
\]

Next, note that \(F = \nabla f\), where \(f(x, y) = xy\). Hence in particular, we have

\[
\int_{\phi} F \cdot ds = \int_{\phi} \nabla f \cdot ds, \quad \int_{\psi} F \cdot ds = \int_{\psi} \nabla f \cdot ds, \quad \int_{\eta} F \cdot ds = \int_{\eta} \nabla f \cdot ds.
\]

Observe now that \(f(1, 1) - f(0, 0) = 1\), so is it a coincidence that

\[
\int_{\phi} \nabla f \cdot ds = \int_{\psi} \nabla f \cdot ds = \int_{\eta} \nabla f \cdot ds = f(1, 1) - f(0, 0),
\]

so that the integrals depend only on the endpoints of the paths? On the other hand, note that \(F\) is the total derivative of \(f\), so (1) is really just a statement like the Fundamental theorem of calculus.

Let us investigate this problem in general. Suppose that \(F\) is a gradient vector field in \(\mathbb{R}^n\), so that there exists a continuously differentiable function \(f(x_1, \ldots, x_n)\) such that \(F = \nabla f\). Suppose that \(\phi : [A, B] \to \mathbb{R}^n\) is a continuously differentiable path. Consider the composition function

\[
g = f \circ \phi : [A, B] \to \mathbb{R}.
\]

By the Chain rule, we have

\[
g'(t) = \left( \frac{\partial f}{\partial x_1}(\phi(t)) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\phi(t)) \right)^T \begin{pmatrix} \phi'_1(t) \\ \vdots \\ \phi'_n(t) \end{pmatrix},
\]
where the right hand side is the matrix product of the total derivatives \((\mathbf{D}f)(\phi(t))\) and \((\mathbf{D}\phi)(t)\). It follows that

\[ g'(t) = (\nabla f)(\phi(t)) \cdot \phi'(t) = F(\phi(t)) \cdot \phi'(t), \]

and so

\[ \int_{\phi} F \cdot d\mathbf{s} = \int_{\mathcal{A}} F(\phi(t)) \cdot \phi'(t) \, dt = \int_{\mathcal{A}} g'(t) \, dt = g(B) - g(A) = f(\phi(B)) - f(\phi(A)) \]

by the Fundamental theorem of calculus applied to the function \(g\). We have proved the following result.

**Theorem 9A.** Suppose that \(F = \nabla f\) is a gradient vector field in \(\mathbb{R}^n\). Then for any continuously differentiable path \(\phi : [A, B] \to \mathbb{R}^n\) such that the composition function \(F \circ \phi : [A, B] \to \mathbb{R}^n\) is continuous, we have

\[ \int_{\phi} F \cdot d\mathbf{s} = f(\phi(B)) - f(\phi(A)). \]

### 9.3. Equivalent Paths

We return to the questions posed by Examples 9.1.4 and 9.2.4.

**Definition.** Suppose that \(\phi : [A_1, B_1] \to \mathbb{R}^n\) and \(\psi : [A_2, B_2] \to \mathbb{R}^n\) are two continuously differentiable paths. Then we say that \(\phi\) and \(\psi\) are equivalent if there exists a continuously differentiable and strictly monotonic function \(h : [A_1, B_1] \to [A_2, B_2]\) such that \(h([A_1, B_1]) = [A_2, B_2]\) and \(\phi = \psi \circ h\). In this case, we say that the function \(h\) defines a change of parameter. Furthermore, we say that the change of parameter is orientation preserving if \(h\) is strictly increasing and orientation reversing if \(h\) is strictly decreasing.

**Remarks.**

1. It is easy to see that if two paths are equivalent, then they have the same curve. If the change of parameter is orientation preserving, then the curve is followed in the same direction. If the change of parameter is orientation reversing, then the curve is followed in different directions.

2. Note that the change of parameter is orientation preserving if and only if \(h'(t) \geq 0\) for every \(t \in [A_1, B_1]\), and orientation reversing if and only if \(h'(t) \leq 0\) for every \(t \in [A_1, B_1]\).

3. Since \(h : [A_1, B_1] \to [A_2, B_2]\) is strictly monotonic and onto, it follows that it has an inverse function \(h^{-1} : [A_2, B_2] \to [A_1, B_1]\). Clearly \(\psi = \phi \circ h^{-1}\). Furthermore, the inverse function is also continuously differentiable.

**Example 9.3.1.** Recall the three distinct paths

\[ \phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t) \]

\[ \psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \]

\[ \eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2) \]

considered in Examples 9.1.4 and 9.2.4. Let us examine first of all \(\psi\) and \(\eta\). The function

\[ h_1 : [0, 1] \to [0, 1] : t \mapsto \sqrt{t} \]

is strictly increasing, and defines an orientation preserving change of parameter with \(\psi = \eta \circ h_1\). Note that the inverse function

\[ h_1^{-1} : [0, 1] \to [0, 1] : t \mapsto t^2 \]

are two continuously differentiable paths. Then we say that \(\phi\) and \(\psi\) are equivalent if there exists a continuously differentiable and strictly monotonic function \(h : [A_1, B_1] \to [A_2, B_2]\) such that \(h([A_1, B_1]) = [A_2, B_2]\) and \(\phi = \psi \circ h\). In this case, we say that the function \(h\) defines a change of parameter. Furthermore, we say that the change of parameter is orientation preserving if \(h\) is strictly increasing and orientation reversing if \(h\) is strictly decreasing.

**Remarks.** (1) It is easy to see that if two paths are equivalent, then they have the same curve. If the change of parameter is orientation preserving, then the curve is followed in the same direction. If the change of parameter is orientation reversing, then the curve is followed in different directions.

(2) Note that the change of parameter is orientation preserving if and only if \(h'(t) \geq 0\) for every \(t \in [A_1, B_1]\), and orientation reversing if and only if \(h'(t) \leq 0\) for every \(t \in [A_1, B_1]\).

(3) Since \(h : [A_1, B_1] \to [A_2, B_2]\) is strictly monotonic and onto, it follows that it has an inverse function \(h^{-1} : [A_2, B_2] \to [A_1, B_1]\). Clearly \(\psi = \phi \circ h^{-1}\). Furthermore, the inverse function is also continuously differentiable.

**Example 9.3.1.** Recall the three distinct paths

\[ \phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, -\sin t) \]

\[ \psi : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \]

\[ \eta : [0, 1] \to \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2) \]

considered in Examples 9.1.4 and 9.2.4. Let us examine first of all \(\psi\) and \(\eta\). The function

\[ h_1 : [0, 1] \to [0, 1] : t \mapsto \sqrt{t} \]

is strictly increasing, and defines an orientation preserving change of parameter with \(\psi = \eta \circ h_1\). Note that the inverse function

\[ h_1^{-1} : [0, 1] \to [0, 1] : t \mapsto t^2 \]
is also strictly increasing, and \( \eta = \psi \circ h_1^{-1} \). Clearly \( \psi \) and \( \eta \) follow the unit circle in the same direction. Consider next \( \phi \) and \( \psi \). The function

\[
h_2 : [0, 2\pi] \to [0, 1] : t \to 1 - \frac{t}{2\pi}
\]

is strictly decreasing, and defines an orientation reversing change of parameter with \( \phi = \psi \circ h_2 \). Note that the inverse function

\[
h_2^{-1} : [0, 1] \to [0, 2\pi] : t \to 2\pi - 2\pi t
\]

is also strictly decreasing, and \( \psi = \phi \circ h_2^{-1} \). Clearly \( \phi \) and \( \psi \) follow the unit circle in opposite directions.

**THEOREM 9B.** Suppose that \( \phi : [A_1, B_1] \to \mathbb{R}^n \) and \( \psi : [A_2, B_2] \to \mathbb{R}^n \) are two equivalent continuously differentiable paths. Then for any real valued function \( f(x_1, \ldots, x_n) \) such that the composition functions \( f \circ \phi : [A_1, B_1] \to \mathbb{R} \) and \( f \circ \psi : [A_2, B_2] \to \mathbb{R} \) are continuous, we have

\[
\int_{\phi} f \, ds = \int_{\psi} f \, ds.
\]

**PROOF.** Since \( \phi \) and \( \psi \) are equivalent, there exists \( h : [A_1, B_1] \to [A_2, B_2] \) such that \( \phi = \psi \circ h \). It follows from the Chain rule that \( \phi'(t) = \psi'(h(t))h'(t) \), and so

\[
\int_{\phi} f \, ds = \int_{A_1}^{B_1} f(h(t)) \| \psi'(h(t)) \| h'(t) \, dt = \int_{A_1}^{B_1} f(\psi(h(t))) \| \psi'(h(t)) \| h'(t) \, dt.
\]

In the orientation preserving case, we have \( h'(t) \geq 0 \) always, and so, with a change of variables \( u = h(t) \), we have

\[
\int_{\phi} f \, ds = \int_{A_1}^{B_1} f(\psi(h(t))) \| \psi'(h(t)) \| h'(t) \, dt = \int_{A_2}^{B_2} f(\psi(u)) \| \psi'(u) \| \, du = \int_{\psi} f \, ds.
\]

In the orientation reversing case, we have \( h'(t) \leq 0 \) always, and so, with a change of variables \( u = h(t) \), we have

\[
\int_{\phi} f \, ds = -\int_{A_1}^{B_1} f(\psi(h(t))) \| \psi'(h(t)) \| h'(t) \, dt = -\int_{B_2}^{A_2} f(\psi(u)) \| \psi'(u) \| \, du
\]

\[
= \int_{A_2}^{B_2} f(\psi(u)) \| \psi'(u) \| \, du = \int_{\psi} f \, ds.
\]

This completes the proof. \( \square \)

**THEOREM 9C.** Suppose that \( \phi : [A_1, B_1] \to \mathbb{R}^n \) and \( \psi : [A_2, B_2] \to \mathbb{R}^n \) are two equivalent continuously differentiable paths. Then for any vector field \( F(x_1, \ldots, x_n) \) such that the composition functions \( F \circ \phi : [A_1, B_1] \to \mathbb{R}^n \) and \( F \circ \psi : [A_2, B_2] \to \mathbb{R}^n \) are continuous, we have

\[
\int_{\phi} F \cdot ds = \pm \int_{\psi} F \cdot ds,
\]

where the equality holds with the + sign if the change of parameter is orientation preserving and with the − sign if the change of parameter is orientation reversing.

**PROOF.** Since \( \phi \) and \( \psi \) are equivalent, there exists \( h : [A_1, B_1] \to [A_2, B_2] \) such that \( \phi = \psi \circ h \). It follows from the Chain rule that \( \phi'(t) = \psi'(h(t))h'(t) \), and so

\[
\int_{\phi} F \cdot ds = \int_{A_1}^{B_1} F(\phi(t)) \cdot \phi'(t) \, dt = \int_{A_1}^{B_1} F(\psi(h(t))) \cdot \psi'(h(t))h'(t) \, dt.
\]
With a change of variables \( u = h(t) \), we have, in the orientation preserving case,

\[
\int_{\phi} F \cdot ds = \int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u) \, du = \int_{\psi} F \cdot ds,
\]
and in the orientation reversing case,

\[
\int_{\phi} F \cdot ds = \int_{B_2}^{A_2} F(\psi(u)) \cdot \psi'(u) \, du = -\int_{A_2}^{B_2} F(\psi(u)) \cdot \psi'(u) \, du = -\int_{\psi} F \cdot ds.
\]

This completes the proof. \( \square \)

**Remark.** Theorems 9B and 9C have natural extensions to the case when the paths are piecewise continuously differentiable. In this case, one can clearly break the paths into continuously differentiable pieces and apply Theorems 9B and 9C to each piece.

### 9.4. Simple Curves

Theorems 9B and 9C demonstrate that integrals over differentiable paths depend only on the (oriented) curves of these paths. It therefore seems natural to try to express the theory in terms of these curves instead of the paths. The purpose of this section is to consider this problem. Before we start, we examine the example below which suggests that some care is required.

**Example 9.4.1.** Consider the curve below with endpoints indicated.

![Diagram of a simple curve with endpoints a and b](image)

Clearly it is not enough to say that a path has initial point \( a \) and terminal point \( b \), since any two paths that trace the curve in the two different ways indicated below are clearly not equivalent.

![Diagram of two paths](image)

To temporarily avoid situations like this, we make the following definition.

**Definition.** By a simple curve \( C \) in \( \mathbb{R}^n \), we mean the image \( C = \phi([A, B]) \) of a piecewise continuously differentiable path \( \phi : [A, B] \to \mathbb{R}^n \) with the property that \( \phi(t_1) \neq \phi(t_2) \) whenever \( A \leq t_1 < t_2 \leq B \), with the possible exception that \( \phi(A) = \phi(B) \) may hold. A simple curve together with a direction is called an oriented simple curve. The function \( \phi \) is called a parametrization of the oriented simple curve \( C \), and the parametrization is said to be orientation preserving if \( \phi \) follows the direction of \( C \), and orientation reversing if \( \phi \) follows the opposite direction of \( C \).

Suppose that \( C \) is an oriented simple curve in \( \mathbb{R}^n \). For any real valued function \( f(x_1, \ldots, x_n) \) continuous on \( C \), we can define

\[
\int_C f \, ds = \int_\phi f \, ds,
\]
where \( \phi \) is any parametrization of \( C \). For any vector field \( F(x_1, \ldots, x_n) \) continuous on \( C \), we can define
\[
\int_C F \cdot ds = \int_\phi F \cdot ds,
\]
where \( \phi \) is any orientation preserving parametrization of \( C \). The integrals
\[
\int_C f \, ds \quad \text{and} \quad \int_C F \cdot ds
\]
are well defined in view of Theorems 9B and 9C respectively.

**Remarks.** (1) Suppose that the oriented simple curve \( C^- \) is obtained from the oriented simple curve \( C \) by taking the opposite orientation. Then
\[
\int_C f \, ds = -\int_{C^-} f \, ds \quad \text{and} \quad \int_C F \cdot ds = -\int_{C^-} F \cdot ds.
\]

(2) The theory can be extended to curves that are not simple, provided that we indicate very carefully how these curves are to be followed, and take note where some parts may be followed more than once. In particular, it is often convenient to break up an oriented curve into several components, each of which is simple. For example, if \( C = C_1 + \ldots + C_k \), where the sum denotes that the oriented curve \( C \) is obtained by following the oriented (simple) curves \( C_1, \ldots, C_k \) one after another, then we have
\[
\int_C f \, ds = \sum_{i=1}^k \int_{C_i} f \, ds \quad \text{and} \quad \int_C F \cdot ds = \sum_{i=1}^k \int_{C_i} F \cdot ds.
\]

In this case, each of \( C_1, \ldots, C_k \) can be parametrized separately.

**Example 9.4.2.** Let \( F(x, y) = (3xy, -y^2) \), and let \( C \) denote the path of the parabola \( y = 2x^2 \) from \((1, 2)\) to \((0, 0)\). Clearly \( \phi : [0, 1] \to \mathbb{R}^2 : t \mapsto (t, 2t^2) \) is an orientation preserving parametrization of \( C^- \), and so
\[
\int_C F \cdot ds = \int_0^1 F(\phi(t)) \cdot \phi'(t) \, dt = \int_0^1 F(t, 2t^2) \cdot (1, 4t) \, dt
\]
\[
= \int_0^1 (6t^3 - 4t^4) \cdot (1, 4t) \, dt = \int_0^1 (6t^3 - 16t^5) \, dt = -\frac{7}{6}.
\]

Hence
\[
\int_C F \cdot ds = \frac{7}{6}.
\]

**Example 9.4.3.** Let \( F(x, y, z) = (2x - y + z, x + y - z^2, 3x - 2y + 4z) \), and let \( C \) denote the circle on the \( xy \)-plane with centre at the origin and radius 3, followed in the anticlockwise direction on the \( xy \)-plane. Clearly \( \phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (3 \cos t, 3 \sin t, 0) \) is an orientation preserving parametrization of \( C \), and so
\[
\int_C F \cdot ds = \int_0^{2\pi} F(\phi(t)) \cdot \phi'(t) \, dt = \int_0^{2\pi} F(3 \cos t, 3 \sin t, 0) \cdot (-3 \sin t, 3 \cos t, 0) \, dt
\]
\[
= \int_0^{2\pi} (6 \cos t - 3 \sin t, 3 \cos t + 3 \sin t, 9 \cos t - 6 \sin t) \cdot (-3 \sin t, 3 \cos t, 0) \, dt
\]
\[
= \int_0^{2\pi} (9 - 9 \sin t \cos t) \, dt = 18\pi.
\]
Example 9.4.4. Let \( F(x, y, z) = (3x^2 + 6y, -14yz, 20xz^2) \), and let \( C \) denote a succession of the straight line segments from \((0, 0, 0)\) to \((1, 0, 0)\) to \((1, 1, 0)\) to \((1, 1, 1)\). Let \( C_1 \) denote the straight line segment from \((0, 0, 0)\) to \((1, 0, 0)\), \( C_2 \) denote the straight line segment from \((1, 0, 0)\) to \((1, 1, 0)\), and \( C_3 \) denote the straight line segment from \((1, 1, 0)\) to \((1, 1, 1)\). Clearly
\[
\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (t, 0, 0), \quad \psi : [0, 1] \to \mathbb{R}^3 : t \mapsto (1, t, 0), \quad \eta : [0, 1] \to \mathbb{R}^3 : t \mapsto (1, 1, t)
\]
are orientation preserving parametrization of \( C_1, C_2, C_3 \) respectively. Hence
\[
\int_{C_1} F \cdot ds = \int_{\phi} F \cdot ds = \int_{0}^{1} F(\phi(t)) \cdot \phi'(t) \, dt = \int_{0}^{1} F(t, 0, 0) \cdot (1, 0, 0) \, dt
\]
\[
= \int_{0}^{1} (3t^2, 0, 0) \cdot (1, 0, 0) \, dt = \int_{0}^{1} 3t^2 \, dt = 1,
\]
\[
\int_{C_2} F \cdot ds = \int_{\psi} F \cdot ds = \int_{0}^{1} F(\psi(t)) \cdot \psi'(t) \, dt = \int_{0}^{1} F(1, t, 0) \cdot (0, 1, 0) \, dt
\]
\[
= \int_{0}^{1} (3 + 6t, 0, 0) \cdot (0, 1, 0) \, dt = \int_{0}^{1} 0 \, dt = 0,
\]
\[
\int_{C_3} F \cdot ds = \int_{\eta} F \cdot ds = \int_{0}^{1} F(\eta(t)) \cdot \eta'(t) \, dt = \int_{0}^{1} F(1, 1, t) \cdot (0, 0, 1) \, dt
\]
\[
= \int_{0}^{1} (9, -14t, 20t^2) \cdot (0, 0, 1) \, dt = \int_{0}^{1} 20t^2 \, dt = \frac{20}{3},
\]
and so
\[
\int_{C} F \cdot ds = \int_{C_1} F \cdot ds + \int_{C_2} F \cdot ds + \int_{C_3} F \cdot ds = \frac{23}{3},
\]
Next, let \( f(x, y, z) = x^2 + y^2 + z^2 \). Then
\[
\int_{C_1} f \, ds = \int_{\phi} f \, ds = \int_{0}^{1} f(\phi(t)) \|\phi'(t)\| \, dt = \int_{0}^{1} f(t, 0, 0) \| (1, 0, 0) \| \, dt = \int_{0}^{1} t^2 \, dt = \frac{1}{3},
\]
\[
\int_{C_2} f \, ds = \int_{\psi} f \, ds = \int_{0}^{1} f(\psi(t)) \|\psi'(t)\| \, dt = \int_{0}^{1} f(1, t, 0) \| (0, 1, 0) \| \, dt = \int_{0}^{1} (1 + t^2) \, dt = \frac{4}{3},
\]
\[
\int_{C_3} f \, ds = \int_{\eta} f \, ds = \int_{0}^{1} f(\eta(t)) \|\eta'(t)\| \, dt = \int_{0}^{1} f(1, 1, t) \| (0, 0, 1) \| \, dt = \int_{0}^{1} (2 + t^2) \, dt = \frac{7}{3},
\]
and so
\[
\int_{C} f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds + \int_{C_3} f \, ds = 4.
\]
1. For each of the following, evaluate the integral $\int f \, ds$:

   a) $f(x, y, z) = \frac{1}{y^2}; \phi : [1, e] \to \mathbb{R}^3 : t \mapsto (\log t, t, 2)$
   
   b) $f(x, y, z) = \cos z; \phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t)$
   
   c) $f(x, y, z) = x \cos z; \phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (t, t^2, 0)$
   
   d) $f(x, y, z) = yz; \phi : [0, 3] \to \mathbb{R}^3 : t \mapsto (t, 2t, 3t)$
   
   e) $f(x, y, z) = z; \phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (t \cos t, t \sin t, t)$
   
   f) $f(x, y, z) = \frac{x + y}{y + z}; \phi : [1, 2] \to \mathbb{R}^3 : t \mapsto (3t, 2t^{3/2}, 3t)$

2. For each of the following, evaluate the integral $\int F \cdot ds$:

   a) $F(x, y, z) = (yz, zx, xy); C$ is the straight line segment from $(1, 0, 0)$ to $(0, 1, 0)$ followed by the straight line segment from $(0, 1, 0)$ to $(0, 0, 1)$.
   
   b) $F(x, y, z) = (3xy, -5z, 10x); \phi : [1, 2] \to \mathbb{R}^3 : t \mapsto (t^2 + 1, 2t^2, t^3)$
   
   c) $F(x, y, z) = (x, y, 2z); \phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, 0)$
   
   d) $F(x, y, z) = (x, 2x, y); \phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (t, t^2, t^3)$
   
   e) $F(x, y, z) = (x^2 + y^2, z, xy); \phi : [0, \pi] \to \mathbb{R}^3 : t \mapsto (\sin t, \cos t, t)$

3. For each of the following, evaluate the integral $\int_C F \cdot ds$:

   a) $F(x, y, z) = (yz, zx, xy); C$ is the straight line segment from $(1, 0, 0)$ to $(0, 1, 0)$ followed by the straight line segment from $(0, 1, 0)$ to $(0, 0, 1)$.
   
   b) $F(x, y, z) = (x^2, -xy, 1); C$ is the parabola $z = x^2$ on the plane $y = 0$ from $(-1, 0, 1)$ to $(1, 0, 1)$.
   
   c) $F(x, y, z) = (x, y, z); C$ is the parabola $y = x^2$ on the plane $z = 0$ from $(-1, 1, 0)$ to $(2, 4, 0)$.

4. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (3x^2, 2xz - y, z)$. 
   
   a) Evaluate the integral $\int_C F \cdot ds$ over each of the paths $\phi$ below:

      (i) $\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (2t, t, 3t)$
      
      (ii) $\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (2t^2, t^2, t^3)$
      
      (iii) $\phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (2t^2, t, 4t^2 - t)$
   
   b) Determine curl $F$.
   
   c) Is $F$ a gradient vector field? Justify your assertion in two different ways, once using part (a), and once using part (b).

5. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y^2z^3 \cos x - 4x^3z, 2yz^3 \sin x, 3y^2z^2 \sin x - x^4)$. 
   
   a) Show that curl $F = 0$ everywhere in $\mathbb{R}^3$.
   
   b) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$ everywhere in $\mathbb{R}^3$.

6. Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (yz, xz + z^2, xy + 2yz)$.
   
   a) Let $\phi : [0, 2\pi] \to \mathbb{R}^3 : t \mapsto (\cos t, \sin t, 1)$. Show that $\int_\phi F \cdot ds = 0$.
   
   b) Show that curl $F = 0$ everywhere in $\mathbb{R}^3$.
   
   c) Find a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = F$ everywhere in $\mathbb{R}^3$.
   
   d) Explain your result in part (a) in terms of part (c), quoting any results that you need.
   
   e) Let $\psi : [0, \pi/2] \to \mathbb{R}^3 : t \mapsto (t \sin t, t^2 \cos^5 t, \cos^4 t + \sin^4 t)$. Evaluate the integral $\int_\psi F \cdot ds$. 

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7. Let \( F(x, y, z) = (z^3 + 2xy, x^2, 3xz^2) \), and consider the integral \( \int_C F \cdot ds \).

a) Evaluate the integral when \( C \) is the square path with vertices \((\pm 1, \pm 1, 0)\) and followed in the anticlockwise direction on the \( xy \)-plane with initial point \((1, 1, 0)\).

b) Find a real valued function \( f(x, y, z) \) such that \( F = \nabla f \).

c) Explain your answer in part (a) in terms of Theorem 9A.

d) Evaluate the integral when \( C \) is the straight line segment from \((1, -2, 1)\) to \((3, 1, 4)\).

e) What is the value of the integral when \( C \) is any curve from \((1, -2, 1)\) to \((3, 1, 4)\)?

8. Evaluate the integral \( \int_C F \cdot ds \) when \( F(x, y, z) = (2xyz, x^2z, x^2y) \) and \( C \) is any simple curve from \((1, 1, 1)\) to \((2, 2, 4)\).

9. It is known that \( \nabla f = F \), where \( F(x, y, z) = (2xyz^2, ze^2, ye^2) \) for every \((x, y, z) \in \mathbb{R}^3\). If \( f(0, 0, 0) = 3 \), what is \( f(1, 1, 2) \)?

10. A path \( \phi : [\theta_1, \theta_2] \to \mathbb{R}^2 \) on the \( xy \)-plane is given in polar coordinates by \( r = r(\theta) \), where \( r(\theta) \) is continuously differentiable in the interval \([\theta_1, \theta_2]\).

a) Determine \( \phi(\theta) \) and \( \phi'(\theta) \) for any \( \theta \in [\theta_1, \theta_2] \).

b) Write the integral \( \int_\phi F \cdot ds \) as an integral over \( \theta \).

c) Find the arc length of the path \( r = 1 + \cos \theta \) where \( \theta \in [0, 2\pi] \).

11. Suppose that \( \phi : [A, B] \to \mathbb{R}^n \) is a continuously differentiable path with arc length \( \ell \). Suppose further that the vector field \( F \) satisfies \( \|F(\mathbf{x})\| \leq M \) for every \( \mathbf{x} \in \phi([A, B]) \). Show that \( \left| \int_\phi F \cdot ds \right| \leq M\ell \).

12. Suppose that \( \phi : [A, B] \to \mathbb{R}^3 \) is a continuously differentiable path. Prove the following:

a) If \( F(\phi(t)) \) is perpendicular to \( \phi'(t) \) for every \( t \in [A, B] \), then \( \int_\phi F \cdot ds = 0 \).

b) If \( F(\phi(t)) \) is in the same direction as \( \phi'(t) \) for every \( t \in [A, B] \), then \( \int_\phi F \cdot ds = \int_\phi \|F\| \cdot ds \).

c) Discuss the case when \( F(\phi(t)) \) is in the opposite direction to \( \phi'(t) \) for every \( t \in [A, B] \).

13. Suppose that \( F \) is a vector field in \( \mathbb{R}^3 \) such that \( \text{curl} \, F = 0 \) everywhere in \( \mathbb{R}^3 \). Follow the steps below to show that there exists a real valued function \( f(x, y, z) \) such that \( \nabla f = F \):

a) Write \( F = (F_1, F_2, F_3) \). Show that \( \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} = \frac{\partial F_1}{\partial x} \).

b) For any \((x, y, z) \in \mathbb{R}^3\), let \( C \) denote a succession of straight line segments from \((0, 0, 0)\) to \((x, 0, 0)\) to \((x, y, 0)\) to \((x, y, z)\). Show that \( \int_C F \cdot ds = \int_0^x F_1(t, 0, 0) \, dt + \int_0^y F_2(x, t, 0) \, dt + \int_0^z F_3(x, y, t) \, dt \).

c) Let \( f(x, y, z) = \int_C F \cdot ds \). Show that \( \frac{\partial f}{\partial z} = F_3 \).

d) Use \( \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \) to show that \( \frac{\partial f}{\partial y} = F_2 \).

e) Use \( \frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \) to show that \( \frac{\partial f}{\partial x} = F_1 \).
14. Consider the vector field
\[ F : A \to \mathbb{R}^3 : (x, y, z) \mapsto \left( \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right), \]
where \( A = \mathbb{R}^3 \setminus \{(0, 0, 0)\} \). Suppose that \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) are two points in \( A \). Let \( C \) be any simple curve from \((x_1, y_1, z_1)\) to \((x_2, y_2, z_2)\).

a) Show that there exists a function \( f : A \to \mathbb{R} \) such that \( \nabla f = F \) everywhere in \( A \).

b) Without using the function \( f \) in part (a) explicitly, use Question 9 to show that the integral \( \int_C F \cdot ds \) depends only on the real numbers \( R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} \) and \( R_2 = \sqrt{x_2^2 + y_2^2 + z_2^2} \).

c) Use Theorem 9A and the function \( f \) in part (a) to draw the same conclusion as in part (b).

15. Let \( F(x, y) = (x^2y, y^2) \), and consider the oriented curve \( C = C_1 + C_2 + C_3 + C_4 \) from \((1, 1)\) to \((4, 1)\) shown in the picture below.

\[ \begin{array}{c}
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\end{array} \]

\[ \begin{array}{c}
\text{a) Find a parametrization } \phi_i \text{ of the oriented line segment } C_i \text{ for each } i = 1, 2, 3, 4. \\
\text{b) Evaluate the integral } \int_{C_i} F \cdot ds \text{ for each } i = 1, 2, 3, 4, \text{ and find the integral } \int_C F \cdot ds.
\end{array} \]