Chapter 7

PATHS

7.1. Introduction

In this chapter, we discuss paths in $\mathbb{R}^n$; in particular, we are interested in paths in $\mathbb{R}^2$ and $\mathbb{R}^3$. Before we give any formal definition, let us consider two examples.

Example 7.1.1. Consider the unit circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in $\mathbb{R}^2$.

At time $t = 0$, a particle at the point $(1, 0) \in C$ starts to move at constant speed along $C$ in the anticlockwise direction and returns for the first time to this initial position at time $t = 2\pi$. It is easy to see that at any time $t \in [0, 2\pi]$, the position of the particle may be given by $\phi(t) = (\cos t, \sin t)$. Here we are interested in the function

$$\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t).$$

Note that $C = \phi([0, 2\pi]) = \{\phi(t) : t \in [0, 2\pi]\}$ is the range of the function $\phi$.

Example 7.1.2. Consider a particle moving away from the origin $0 = (0, 0, 0)$ at time $t = 0$ in the direction of the unit vector $u \in \mathbb{R}^3$ with constant acceleration $a$, and hence speed $ta$ at any given time
$t \geq 0$. In this case, the distance of the particle from the origin at time $t$ is given by $\frac{1}{2}t^2a$, and so its position is given by $\phi(t) = \frac{1}{2}t^2au$. Suppose that we trace the movement of this particle from $t = 0$ to $t = T$. Then we are interested in the function

$$\phi : [0, T] \rightarrow \mathbb{R}^3 : t \mapsto \frac{1}{2}t^2au. \quad (2)$$

The range of this function is given by $\phi([0, T]) = \left\{ \frac{1}{2}t^2au : t \in [0, T] \right\}$, and is a line segment joining the origin $0$ and the point $\frac{1}{2}T^2au$.

Note that the functions (1) and (2) above do not just trace out curves. They also give the position of the particles at any time within the time interval.

**Definition.** By a path in $\mathbb{R}^n$, we mean a function of the type

$$\phi : [A, B] \rightarrow \mathbb{R}^n,$$

where $A, B \in \mathbb{R}$ and $A < B$. The range

$$\phi([A, B]) = \{ \phi(t) : t \in [A, B] \} \subseteq \mathbb{R}^n$$

of the function $\phi$ is called a curve, with initial point $\phi(A)$ and terminal point $\phi(B)$. Suppose that for every $t \in [A, B]$, we have $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, where $\phi_1(t), \ldots, \phi_n(t) \in \mathbb{R}$. Then the functions $\phi_i : [A, B] \rightarrow \mathbb{R}$, where $i = 1, \ldots, n$, are called the components of the path $\phi$.

**Remarks.**

1. We usually write $\phi(t) = (x(t), y(t))$ and $\phi(t) = (x(t), y(t), z(t))$ in the cases $n = 2$ and $n = 3$ respectively.

2. Note the distinction between a path and a curve. Quite often, distinct paths may share the same curve. For example, the three distinct paths

$$\begin{align*}
\phi : [0, 2\pi] &\rightarrow \mathbb{R}^2 : t \mapsto (\cos t, \sin t) \\
\psi : [0, 1] &\rightarrow \mathbb{R}^2 : t \mapsto (\cos 2\pi t, \sin 2\pi t) \\
\eta : [0, 1] &\rightarrow \mathbb{R}^2 : t \mapsto (\cos 2\pi t^2, \sin 2\pi t^2)
\end{align*}$$

satisfy $\phi([0, 2\pi]) = \psi([0, 1]) = \eta([0, 1]) = C$, the unit circle in $\mathbb{R}^2$.

3. Often, we refer to the path $\phi(t)$ without specifying the domain of definition of the function $\phi$. This is a convenient abuse of rigour.

**Example 7.1.3.** Consider a circular disc of radius $r$ standing on a level surface. Let $C$ denote the centre of the disc, and let $P$ denote a fixed point on the rim of the disc. Suppose that at time $t = 0$, the point $P$ touches the surface, and is therefore directly below the point $C$. For convenience, let us assume that this point where the disc touches the surface at time $t = 0$ is the origin $(0, 0)$.

The disc now starts rolling to the right at constant speed $v$. We now wish to describe the path taken by the point $P$. Clearly the point $C$ is at position $(0, r)$ at time $t = 0$. Its position at time $t$ is given by $(tv, r)$. Note next that the circumference of the disc is $2\pi r$, and so the disc will complete one revolution at time $t = 2\pi r/v$. It follows that the angular speed of the disc is $v/r$. Now let $\psi(t)$ denote the relative
position of $P$ with respect to $C$. Clearly $P$ rotates around $C$ in a clockwise direction with angular speed $v/r$, so it follows that

$$
\psi(t) = \left( r \cos \left( -\frac{vt}{r} + \theta \right), r \sin \left( -\frac{vt}{r} + \theta \right) \right),
$$

where $\theta \in \mathbb{R}$ is a constant. Clearly $\psi(0) = (0, -r)$, so that $\cos \theta = 0$ and $\sin \theta = -1$, whence $\theta = -\pi/2$. Hence

$$
\psi(t) = \left( r \cos \left( \frac{vt}{r} + \frac{\pi}{2} \right), -r \sin \left( \frac{vt}{r} + \frac{\pi}{2} \right) \right) = \left( -r \sin \frac{vt}{r}, -r \cos \frac{vt}{r} \right).
$$

It follows that the actual position of $P$ at time $t$ is given by

$$
\phi(t) = (tv, r) + \psi(t) = \left( tv - r \sin \frac{vt}{r}, r - r \cos \frac{vt}{r} \right).
$$

Suppose that $v = r = 1$. Then

$$
\phi(t) = (t - \sin t, 1 - \cos t).
$$

Clearly the point $P$ touches the surface when $t = 2k\pi$, where $k$ is a non-negative integer. The image curve of the path

$$
\phi : [A, B] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)
$$

is called a cycloid. Note that we have not specified the range for $t$ in our discussion. We can consider any interval $[A, B] \subseteq \mathbb{R}^2$, although to get a full picture, the interval should have length at least $2\pi$. A picture of the path

$$
\phi : [0, 4\pi] \to \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t)
$$

is given below.

![Cycloid](image_url)

### 7.2. Differentiable Paths

**Definition.** We say that a path $\phi : [A, B] \to \mathbb{R}^n$ is differentiable if the limit

$$
\lim_{h \to 0} \frac{\phi(t + h) - \phi(t)}{h}
$$

exists for every $t \in [A, B]$, with the obvious restriction to one-sided limits at the endpoints of the interval $[A, B]$. In this case, the vector

$$
\phi'(t) = \frac{d}{dt} \phi(t) = \lim_{h \to 0} \frac{\phi(t + h) - \phi(t)}{h}
$$

is called the velocity vector of the path $\phi$, and the quantity $\|\phi'(t)\|$ is called the speed of the path $\phi$. 

*Chapter 7: Paths*
REMARKS. (1) Note that we have borrowed some terminology from physics. This is entirely natural, as this area of mathematics is, to a large extent, motivated by the study of various problems in physics.

(2) Note that if the path is given by \( \phi(t) = (\phi_1(t), \ldots, \phi_n(t)) \), then the velocity vector is given by \( \phi'(t) = (\phi'_1(t), \ldots, \phi'_n(t)) \) and the speed is given by \( \|\phi(t)\| = (|\phi'_1(t)|^2 + \ldots + |\phi'_n(t)|^2)^{1/2} \).

(3) Note the special notation in the cases \( n = 2 \) and \( n = 3 \).

(4) The velocity vector \( \phi'(t) \) is a vector tangent to the path \( \phi(t) \) at time \( t \). If \( C \) is the curve of the path \( \phi(t) \) and \( \phi'(t) \neq 0 \), then \( \phi'(t) \) is a vector tangent to the curve \( C \) at the point \( \phi(t) \in C \).

EXAMPLE 7.2.1. For the cycloid \( \phi(t) = (t - \sin t, 1 - \cos t) \) described in Example 7.1.3, the velocity vector is given by \( \phi'(t) = (1 - \cos t, \sin t) \). Note that \( 1 - \cos t = 0 \) implies that \( \sin t = 0 \), so the velocity is never vertical. The speed of the path is
\[
\|\phi'(t)\| = ((1 - \cos t)^2 + \sin^2 t)^{1/2} = (2 - 2 \cos t)^{1/2}.
\]
This is minimum and zero when \( \cos t = 1 \), when the point \( P \) touches the surface. The speed is maximum when \( \cos t = -1 \), when the point \( P \) is at the maximum height.

EXAMPLE 7.2.2. To study the path \( \phi(t) = (\cos t, \sin t, t) \) in \( \mathbb{R}^3 \), we first of all consider the first two components, and study the path \( \psi(t) = (\cos t, \sin t) \) in \( \mathbb{R}^2 \). This path describes a circle on the plane, followed in the anticlockwise direction. The third component \( t \) describes an increase in height with time if we think of the third component as the vertical component. It follows that if we consider the cylinder \( x^2 + y^2 = 1 \) in \( \mathbb{R}^3 \), then the path \( \phi(t) \) wraps round this cylinder in an anticlockwise direction with the third component increasing if we look from above. The curve of the path \( \phi(t) \) is called a helix.

\[\text{Example diagram here}\]

The path has velocity vector \( \phi'(t) = (-t, \sin t, 1) \) and speed \( \|\phi'(t)\| = (\sin^2 t + \cos^2 t + 1)^{1/2} = \sqrt{2} \), so the path has constant speed.

Suppose that \( \phi(t) \) is a differentiable path. We have already indicated that if \( \phi'(t_0) \neq 0 \), then it is a vector tangent to the path at the point \( \phi(t_0) \). It follows immediately that

**THEOREM 7A.** Suppose that \( \phi(t) \) is a differentiable path in \( \mathbb{R}^n \). Then the tangent line to the path at the point \( \phi(t_0) \) is given by
\[
L(\lambda) = \phi(t_0) + \lambda \phi'(t_0),
\]
provided that \( \phi'(t_0) \neq 0 \).
Example 7.2.3. The equation of the tangent line to the helix \( \phi(t) = (\cos t, \sin t, t) \) at the point \( \phi(t_0) \) is given by

\[
L(\lambda) = \phi(t_0) + \lambda \phi'(t_0) = (\cos t_0, \sin t_0, t_0) + \lambda(-\sin t_0, \cos t_0, 1).
\]

Suppose that \( t_0 = 2\pi \). Then \( \phi(2\pi) = (1, 0, 2\pi) \), and the tangent line becomes \( L(\lambda) = (1, 0, 2\pi) + \lambda(0, 1, 1) \).

Writing \( L(\lambda) = (x, y, z) \), we have \( x = 1 \), \( y = \lambda \) and \( z = 2\pi + \lambda \). It follows that the tangent line to the helix at the point \( (1, 0, 2\pi) \) is given by \( x = 1 \) and \( z = y + 2\pi \). Try to visualize this from the picture in Example 7.2.2.

Example 7.2.4. The equation of the tangent line to the cycloid \( \phi(t) = (t - \sin t, 1 - \cos t) \) at the point \( \phi(t_0) \) is given by

\[
L(\lambda) = \phi(t_0) + \lambda \phi'(t_0) = (t_0 - \sin t_0, 1 - \cos t_0) + \lambda(1 - \cos t_0, \sin t_0).
\]

Suppose that \( t_0 = 2\pi \). Then \( \phi(2\pi) = (2\pi, 0) \), and \( L(\lambda) = (2\pi, 0) + \lambda(0, 0) = (2\pi, 0) \), clearly not the equation of a line. Observe that since \( \phi'(2\pi) = (0, 0) \), Theorem 7A does not apply in this case. In fact, the tangent line is vertical.

Example 7.2.5. Let us return to the helix discussed in Examples 7.2.2 and 7.2.3. Suppose that a particle follows the helix from \( t = 0 \) to \( t = 2\pi \) and then flies off at constant velocity on a tangent at \( t = 2\pi \). We wish to determine the position of the particle at \( t = 4\pi \). Note that the particle is at position \( \phi(2\pi) = (1, 0, 2\pi) \) when \( t = 2\pi \), with tangential velocity \( \phi'(2\pi) = (0, 1, 1) \). It follows that its position at \( t = 4\pi \) must be given by

\[
\phi(2\pi) + (4\pi - 2\pi)\phi'(2\pi) = (1, 0, 2\pi) + 2\pi(0, 1, 1) = (1, 2\pi, 4\pi).
\]

Example 7.2.6. Consider the hypocycloid of four cusps

\[
\phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos^3 t, \sin^3 t).
\]

This path has velocity vector

\[
\phi'(t) = (-3 \cos^2 t \sin t, 3 \sin^2 t \cos t)
\]

and speed

\[
\|\phi'(t)\| = (9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t)^{1/2} = 3|\cos t \sin t|.
\]

Note that while the hypocycloid is a differentiable path, its curve has cusps. Note, however, that the velocity and speed are zero at these cusps.

We state without proof the following two theorems. The proofs are not difficult, and follow by applying the usual differentiation rules to the components.
THEOREM 7B. Suppose that \( \phi(t) \) and \( \psi(t) \) are differentiable paths in \( \mathbb{R}^n \). Suppose further that \( a(t) \) and \( b(t) \) are differentiable real valued functions. Then
\[
\begin{align*}
(a) \quad & \frac{d}{dt}(\phi(t) + \psi(t)) = \phi'(t) + \psi'(t); \\
(b) \quad & \frac{d}{dt}(a(t)\phi(t)) = a(t)\phi'(t) + a'(t)\phi(t); \\
(c) \quad & \frac{d}{dt}(\phi(t) \cdot \psi(t)) = \phi(t) \cdot \psi'(t) + \phi'(t) \cdot \psi(t); \quad \text{and} \\
(d) \quad & \frac{d}{dt}(a(\phi(t))) = a'(t)\phi'(a(t)).
\end{align*}
\]

The above represent the sum rule, scalar multiplication rule, dot product rule and chain rule respectively. Note also the vector product rule below which is valid only in \( \mathbb{R}^3 \).

THEOREM 7C. Suppose that \( \phi(t) \) and \( \psi(t) \) are differentiable paths in \( \mathbb{R}^3 \). Then
\[
\frac{d}{dt}(\phi(t) \times \psi(t)) = \phi(t) \times \psi'(t) + \phi'(t) \times \psi(t).
\]

7.3. Arc Length

In this section, we are interested in calculating the length of the curve followed by a path. To motivate this, note that the speed \( \|\phi'(t)\| \) of a path \( \phi(t) \) is the rate of change of distance with respect to time.

DEFINITION. Suppose that \( \phi : [A, B] \to \mathbb{R}^n \) is a differentiable path. The velocity differential is given by
\[
\frac{ds}{dt} = \phi'(t) \, dt = (\phi'_1(t), \ldots, \phi'_n(t)) \, dt.
\]

The corresponding arc length differential is given by
\[
\frac{ds}{dt} = \|\phi'(t)\| \, dt = (\|\phi'_1(t)\|^2 + \ldots + \|\phi'_n(t)\|^2)^{1/2} \, dt.
\]

REMARKS. (1) The velocity differential describes an infinitesimal displacement of a particle following the path \( \phi \). The arc length differential describes the magnitude of this infinitesimal displacement.

(2) In \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), we have velocity differential
\[
\frac{ds}{dt} = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \, dt \quad \text{and} \quad \frac{ds}{dt} = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \, dt
\]

and arc length differential
\[
\frac{ds}{dt} = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right)^{1/2} \, dt \quad \text{and} \quad \frac{ds}{dt} = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \right)^{1/2} \, dt
\]

respectively.
DEFINITION. Suppose that \( \phi : [A, B] \rightarrow \mathbb{R}^n \) is a continuously differentiable path. Then the quantity

\[
\ell = \int_A^B \| \phi'(t) \| \, dt
\]

is called the arc length of the path \( \phi \).

REMARK. Note that if \( \phi(t) = (\phi_1(t), \ldots, \phi_n(t)) \), then

\[
\ell = \int_A^B (|\phi'_1(t)|^2 + \ldots + |\phi'_n(t)|^2)^{1/2} \, dt.
\]

EXAMPLE 7.3.1. The cycloid \( \phi : [0, 2\pi] \rightarrow \mathbb{R}^2 : t \mapsto (t - \sin t, 1 - \cos t) \) has arc length

\[
\ell = \int_0^{2\pi} \| \phi'(t) \| \, dt = \int_0^{2\pi} (2 - 2 \cos t)^{1/2} \, dt = 8.
\]

EXAMPLE 7.3.2. The helix \( \phi : [0, 2\pi] \rightarrow \mathbb{R}^3 : t \mapsto (\cos t, \sin t, t) \) has arc length

\[
\ell = \int_0^{2\pi} \| \phi'(t) \| \, dt = \int_0^{2\pi} \sqrt{2} \, dt = 2\pi\sqrt{2}.
\]

EXAMPLE 7.3.3. The hypocycloid of four cusps \( \phi : [0, 2\pi] \rightarrow \mathbb{R}^2 : t \mapsto (\cos 3t, \sin 3t) \) has arc length

\[
\ell = \int_0^{2\pi} \| \phi'(t) \| \, dt = \int_0^{2\pi} 3|\cos t \sin t| \, dt = 12 \int_0^{\pi/2} \sin t \cos t \, dt = 6.
\]
PROBLEMS FOR CHAPTER 7

1. Sketch the curve of each of the following paths:
   a) \( \phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\sin t, 3 \cos t) \)
   b) \( \phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (2t + 1, t + 2, t) \)

2. For each of the following paths, find the equation of the tangent line at the point \( \phi(t_0) \):
   a) \( \phi : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (e^t, \cos t) \)
   b) \( \phi : [0, 1] \to \mathbb{R}^3 : t \mapsto (t^3, t^2, t) \)

3. A particle follows the path \( \phi(t) = (\sin e^t, t, 4 - t^3) \) in \( \mathbb{R}^3 \) from time \( t = 0 \) to time \( t = 1 \), and then flies off at a tangent at constant velocity. Determine its position at time \( t = 3 \).

4. Consider the path \( \phi : [0, 1] \to \mathbb{R}^2 : t \mapsto (t \cos t, t \sin t) \).
   a) Determine the velocity vector \( \phi'(\pi/6) \).
   b) Find the equation of the tangent line to the path at \( \phi(\pi/6) \), if it exists.
   c) Determine the speed \( \|\phi'(t)\| \) for every \( t \in [0, 1] \).
   d) Determine the arc length of the path \( \phi \).
      [Hint: You may need a substitution and integration by parts.]

5. For each of the following paths, determine its arc length:
   a) \( \phi : [0, \pi] \to \mathbb{R}^3 : t \mapsto (t, t \cos t, t \sin t) \)
   b) \( \phi : [0, 4\pi] \to \mathbb{R}^3 : t \mapsto \begin{cases} 
   (2 \cos t, t, 2 \sin t) & \text{if } t \in [0, 2\pi] \\
   (2, t, t - 2\pi) & \text{if } t \in [2\pi, 4\pi]
   \end{cases} \)