Chapter 8

LINEAR TRANSFORMATIONS
ON HILBERT SPACES

8.1. Adjoint Transformations

We begin with a result which is a consequence of the Riesz-Fréchet theorem first studied in Section 6.3.

**Theorem 8A.** Suppose that \( V \) and \( W \) are Hilbert spaces over \( F \). For every linear transformation \( T \in B(V,W) \), there exists a unique linear transformation \( T^* \in B(W,V) \) such that

\[
\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for every } x \in V \text{ and } y \in W.
\]

**Remark.** Note that in the above, \( \langle T(x), y \rangle \) is an inner product in the Hilbert space \( W \), while \( \langle x, T^*(y) \rangle \) is an inner product in the Hilbert space \( V \).

**Definition.** Suppose that \( V \) and \( W \) are Hilbert spaces over \( F \). The unique linear transformation \( T^* \in B(W,V) \) satisfying the conclusion of Theorem 8A is called the adjoint transformation of the linear transformation \( T \in B(V,W) \).

**Proof of Theorem 8A.** Suppose that \( y \in W \) is fixed. It is easy to check that the mapping \( S : V \to F \), given for every \( x \in V \) by \( S(x) = \langle T(x), y \rangle \), is a linear functional on \( V \). Furthermore, we have

\[
|S(x)| = |\langle T(x), y \rangle| \leq \|T(x)\| \|y\| \leq \|T\| \|x\| \|y\| = (\|T\| \|y\|) \|x\| \quad \text{for every } x \in V,
\]

so that \( S : V \to F \) is a bounded, and continuous, linear functional on \( V \). It follows from the Riesz-Fréchet theorem that there exists a unique \( u \in V \) such that \( S(x) = \langle x, u \rangle \) for every \( x \in V \). Write \( u = T^*(y) \). Then \( T^* : W \to V \) is a mapping satisfying

\[
\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for every } x \in V \text{ and } y \in W.
\]
Next we show that $T^* \in B(W, V)$. Suppose first of all that $y, z \in W$ and $c \in \mathbb{F}$. Then for every $x \in V$, we have
\[
\langle x, T^*(y + z) \rangle = \langle T(x), y + z \rangle = \langle T(x), y \rangle + \langle T(x), z \rangle \\
= \langle x, T^*(y) \rangle + \langle x, T^*(z) \rangle = \langle x, T^*(y) + T^*(z) \rangle
\]
and
\[
\langle x, T^*(cy) \rangle = \langle T(x), cy \rangle = \tau(T(x), y) = \tau(x, T^*(y)) = \langle x, cT^*(y) \rangle,
\]
so that $T^*(y + z) = T^*(y) + T^*(z)$ and $T^*(cy) = cT^*(y)$ by Theorem 4B. It follows that $T^*: W \to V$ is a linear transformation. Furthermore, for every $y \in W$, we have
\[
\|T^*(y)\|^2 = \langle T^*(y), T^*(y) \rangle = \langle T(T^*(y)), y \rangle \leq \|T(T^*(y))\| \|y\| \leq \|T\| \|T^*(y)\| \|y\|.
\]
Suppose that $\|T^*(y)\| > 0$. Then dividing the above by $\|T^*(y)\|$, we obtain $\|T^*(y)\| \leq \|T\| \|y\|$. Note that this last inequality is satisfied trivially if $\|T^*(y)\| = 0$. It follows that
\[
\|T^*(y)\| \leq \|T\| \|y\| \quad \text{for every } y \in W,
\]
and so $T^*: W \to V$ is bounded, whence $T^* \in B(W, V)$. Finally, suppose that $T_1, T_2 \in B(W, V)$ satisfy
\[
\langle T(x), y \rangle = \langle x, T_1(y) \rangle = \langle x, T_2(y) \rangle \quad \text{for every } x \in V \text{ and } y \in W.
\]
Then it follows from Theorem 4B that $T_1(y) = T_2(y)$ for every $y \in W$, so that $T_1 = T_2$. The uniqueness of $T^* \in B(W, V)$ follows immediately. \(\Box\)

**Example 8.1.1.** Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the vector space $L^2[a, b]$ of all complex valued Lebesgue measurable functions that are square integrable on $[a, b]$. We know that the norm
\[
\|f\| = \left( \int_a^b |f(t)|^2 \, dt \right)^{1/2},
\]
given in Example 7.1.3, is in fact induced by the inner product
\[
\langle f, g \rangle = \left( \int_a^b f(t) g(t) \, dt \right)^{1/2}.
\]
Let $\phi \in C[a, b]$ be chosen and fixed, and consider the bounded linear operator $T : L^2[a, b] \to L^2[a, b]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[a, b]$ is defined by
\[
(T(f))(t) = \phi(t) f(t) \quad \text{for every } t \in [a, b],
\]
as discussed in Example 7.1.3. It follows from Theorem 8A that the adjoint operator $T^*$ satisfies
\[
\langle T(f), g \rangle = \langle f, T^*(g) \rangle \quad \text{for every } f, g \in L^2[a, b].
\]
In other words, we must have
\[
\int_a^b \phi(t) f(t) \overline{g(t)} \, dt = \int_a^b f(t) \overline{(T^*(g))(t)} \, dt \quad \text{for every } f, g \in L^2[a, b].
\]
Clearly
\[
(T^*(g))(t) = \overline{\phi(t) g(t)} \quad \text{for every } t \in [a, b]
\]
would be sufficient. Hence by uniqueness, the adjoint operator $T^*: L^2[a, b] \to L^2[a, b]$ is given for every $g \in L^2[a, b]$ by this.
Example 8.1.2. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Consider the vector spaces $L^2[a, b]$ and $L^2[c, d]$. We know that the respective norms

$$
\|f\| = \left( \int_a^b |f(t)|^2 \, dt \right)^{1/2} \quad \text{and} \quad \|h\| = \left( \int_c^d |h(s)|^2 \, ds \right)^{1/2},
$$
given in Example 7.1.4, are in fact induced by the respective inner products

$$
\langle f, g \rangle = \left( \int_a^b f(t)\overline{g(t)} \, dt \right)^{1/2} \quad \text{and} \quad \langle h, k \rangle = \left( \int_c^d h(s)\overline{k(s)} \, ds \right)^{1/2}.
$$

Let $\phi : [c, d] \times [a, b] \rightarrow \mathbb{C}$ be a fixed continuous function, and consider the bounded linear transformation $T : L^2[a, b] \rightarrow L^2[c, d]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[c, d]$ is defined by

$$
(T(f))(s) = \int_a^b \phi(s,t) f(t) \, dt \quad \text{for every } s \in [c, d],
$$
as discussed in Example 7.1.4. It follows from Theorem 8A that the adjoint operator $T^*$ satisfies

$$
\langle T(f), k \rangle = \langle f, T^*(k) \rangle \quad \text{for every } f \in L^2[a, b] \text{ and } k \in L^2[c, d].
$$

In other words, we must have

$$
\int_c^d \left( \int_a^b \phi(s,t) f(t) \, dt \right) \overline{k(s)} \, ds = \int_a^b f(t) (T^*(k))(t) \, dt \quad \text{for every } f \in L^2[a, b] \text{ and } k \in L^2[c, d].
$$

By Fubini’s theorem, clearly

$$
(T^*(k))(t) = \int_c^d \phi(s,t) k(s) \, ds \quad \text{for every } t \in [a, b]
$$

would be sufficient. Hence by uniqueness, the adjoint transformation $T^* : L^2[c, d] \rightarrow L^2[a, b]$ is given for every $k \in L^2[c, d]$ by this.

8.2. Hermitian Operators

We conclude our discussion by studying a special type of adjoint operators.

Definition. Suppose that $V$ is a Hilbert space over $\mathbb{F}$. A linear operator $T \in L(V)$ is said to be self-adjoint or Hermitian if $T^* = T$.

Example 8.2.1. Suppose that $a, b \in \mathbb{R}$ and $a < b$. Consider the Hilbert space $L^2[a, b]$ of all complex valued Lebesgue measurable functions that are square integrable on $[a, b]$, as discussed in Example 8.1.1. Let $\phi \in C[a, b]$ be chosen and fixed. For the bounded linear operator $T : L^2[a, b] \rightarrow L^2[a, b]$, where for every $f \in L^2[a, b]$, the function $T(f) \in L^2[a, b]$ is defined by $(T(f))(t) = \phi(t)f(t)$ for every $t \in [a, b]$, we have shown earlier that the adjoint operator $T^* : L^2[a, b] \rightarrow L^2[a, b]$ is given for every $g \in L^2[a, b]$ by $(T^*(g))(t) = \overline{\phi(t)}g(t)$ for every $t \in [a, b]$. Hence $T : L^2[a, b] \rightarrow L^2[a, b]$ is Hermitian if $\phi \in C[a, b]$ is real valued.

The following result gives a technique for finding the norm of a Hermitian operator.
THEOREM 8B. Suppose that $V$ is a Hilbert space over $\mathbb{F}$. Suppose further that $T \in B(V)$ is an Hermitian operator. Then

$$\|T\| = \sup_{x \in V, \|x\| = 1} |\langle T(x), x \rangle|.$$ 

PROOF. For every $x \in V$ satisfying $\|x\| = 1$, we have

$$|\langle T(x), x \rangle| \leq \|T(x)\| \|x\| \leq \|T\| \|x\|^2 = \|T\|,$$

so that

$$\|T\| \geq \sup_{x \in V, \|x\| = 1} |\langle T(x), x \rangle|.$$

To prove the opposite inequality, let

$$M = \sup_{x \in V, \|x\| = 1} |\langle T(x), x \rangle|.$$

For any non-zero vector $u \in V$, the vector $u/\|u\|$ has norm 1. It follows easily from linearity that

$$|\langle T(u), u \rangle| \leq M \|u\|^2 \quad \text{for every } u \in V.$$

For every $x, y \in V$, noting that $T^* = T$, it is not difficult to check that

$$\langle T(x + y), x + y \rangle = \langle T(x) + T(y), x + y \rangle$$

$$= \langle T(x), x \rangle + \langle T(x), y \rangle + \langle T(y), x \rangle + \langle T(y), y \rangle$$

$$= \langle T(x), x \rangle + \langle T(x), y \rangle + \langle y, T^*(x) \rangle + \langle T(y), y \rangle$$

$$= \langle T(x), x \rangle + \langle T(x), y \rangle + \overline{\langle T(x), y \rangle} + \langle T(y), y \rangle$$

$$= \langle T(x), x \rangle + 2\Re \langle T(x), y \rangle + \langle T(y), y \rangle,$$

and similarly

$$\langle T(x - y), x - y \rangle = \langle T(x), x \rangle - 2\Re \langle T(x), y \rangle + \langle T(y), y \rangle,$$

and so

$$4\Re \langle T(x), y \rangle = \langle T(x + y), x + y \rangle - \langle T(x - y), x - y \rangle$$

$$\leq M(\|x + y\|^2 + \|x - y\|^2)$$

$$= 2M(\|x\|^2 + \|y\|^2),$$

the last step as a consequence of the Parallelogram law. We can replace $x$ by $\lambda x$, where $\lambda \in \mathbb{F}$ satisfies $|\lambda| = 1$ and $\Re \langle T(\lambda x), y \rangle = |\langle T(x), y \rangle|$. Then

$$|\langle T(x), y \rangle| \leq \frac{1}{2} M(\|x\|^2 + \|y\|^2) \quad \text{for every } x, y \in V.$$

Suppose first of all that $T(x) \neq 0$. Then taking

$$y = \frac{\|x\|}{\|T(x)\|} T(x),$$

we have $\|y\| = \|x\|$ and

$$\left| \frac{\|x\|}{\|T(x)\|} |\langle T(x), T(x) \rangle| \right| \leq M \|x\|^2,$$

so that $\|T\| \leq M \|x\|$.

The last inequality holds trivially if $T(x) = 0$, and therefore holds for every $x \in V$. It follows that $\|T\| \leq M$ as required. $\Box$
PROBLEMS FOR CHAPTER 8

1. Suppose that $V$ and $W$ are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$.
   a) Show that $(y, (T^*)^*(x)) = (y, T(x))$ for every $x \in V$ and $y \in W$, using condition (IP1) twice. Explain why this implies that $(T^*)^* = T$.
   b) Deduce that $\|T^*\| = \|T\|$. You may wish to study the proof of Theorem 8A for some useful information.
   c) Show that $\|T(x)\| \leq \|T^*T\|\|x\|$ for every $x \in V$. Explain why this implies the inequality $\|T\|^2 \leq \|T^*T\|$.
   d) Deduce that $\|T^*T\| = \|T\|^2$.

2. Suppose that $V$, $W$, and $U$ are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$ and $S \in B(W,U)$. Show that $(ST)^* = T^*S^*$.

3. Suppose that $V$ and $W$ are Hilbert spaces over $\mathbb{F}$. Show that for every $c, a \in \mathbb{F}$ and $T, S \in B(V,W)$, we have $(cT + aS)^* = cT^* + aS^*$.

4. Suppose that $V$ and $W$ are Hilbert spaces over $\mathbb{F}$. Show that the function $f : B(V,W) \rightarrow B(W,V)$, defined for every $T \in B(V,W)$ by $f(T) = T^*$, is continuous in $B(V,W)$.
   [HINT: Show that $\|f(T) - f(S)\| = \|T - S\|$ for every $T, S \in B(V,W)$].

5. Suppose that $V$ is a complex Hilbert space, and that $x_1, x_2 \in V$ are fixed. Consider the bounded linear operator $T : V \rightarrow V$, where $T(x) = (x, x_1)x_2$ for every $x \in V$. Show that the adjoint operator $T^* : V \rightarrow V$ is given by $T^*(y) = (y, x_2)x_1$ for every $y \in V$.

6. Consider the vector space $\ell^2$ of all square summable infinite sequences of complex numbers, with inner product

   $$\langle x, y \rangle = \left( \sum_{i=1}^{\infty} x_i\overline{y_i} \right)^{1/2}.
   $$

   For each of the given bounded linear operators $T : \ell^2 \rightarrow \ell^2$, find the adjoint operator $T^* : \ell^2 \rightarrow \ell^2$:
   a) $T(x) = (0, x_1, x_2, x_3, \ldots)$ for every $x = (x_1, x_2, x_3, \ldots)$, as discussed in Example 7.1.6.
   b) $T(x) = (0, 2x_1, 2x_2, 2x_3, x_4, \ldots)$ for every $x = (x_1, x_2, x_3, x_4, \ldots)$, as discussed in Problem 1 in Chapter 7.

7. Suppose that $(x_n)_{n \in \mathbb{N}}$ is an orthonormal basis in a Hilbert space $V$ over $\mathbb{C}$, and that $(c_n)_{n \in \mathbb{N}}$ is a fixed bounded sequence of complex numbers. Consider the bounded linear operator $T : V \rightarrow V$ such that $T(x_n) = c_nx_n$ for every $n \in \mathbb{N}$, discussed in Problem 3 in Chapter 7. Find the adjoint operator $T^* : V \rightarrow V$.

8. Suppose that $V$ and $W$ are Hilbert spaces over $\mathbb{F}$, and that $T \in B(V,W)$. Suppose also that $R(T)$ and $R(T^*)$ denote respectively the range of the linear transformations $T : V \rightarrow W$ and $T^* : W \rightarrow V$.
   a) Show that $(x, z) = 0$ for every $x \in \ker(T)$ and $z \in R(T^*)$.
   b) Deduce that $\ker(T) \subseteq (R(T^*))^\perp$.
   c) Show that $(T(u), T(u)) = 0$ for every $u \in (R(T^*))^\perp$.
   d) Deduce that $(R(T^*))^\perp \subseteq \ker(T)$.
   e) It follows from parts (b) and (d) that $\ker(T) = (R(T^*))^\perp$. Use this and Problem 1 to show that $\ker(T^*) = (R(T))^\perp$.

9. Suppose that $V$ is a Hilbert space over $\mathbb{F}$. Suppose further that $T \in B(V)$ is invertible, so that $TT^{-1} = T^{-1}T = I$, where $I \in B(V)$ is the identity linear operator.
   a) Show that $T^* = I$.
   b) By studying the adjoint of the equation $TT^{-1} = T^{-1}T = I$, show that $T^*$ is invertible, with inverse $(T^*)^{-1} = (T^{-1})^*$. 

\[ \text{Chapter 8 : Linear Transformations on Hilbert Spaces} \]
10. Suppose that $V$ is a Hilbert space over $F$. Suppose further that $\mathcal{H}$ is the subset of all Hermitian operators in $B(V)$.
   a) Show that $cT + aS \in \mathcal{H}$ for every $c, a \in \mathbb{R}$ and $T, S \in \mathcal{H}$.
   b) Show that $\mathcal{H}$ is a closed subset of $B(V)$.
      [HINT: Use Problem 4.]

11. Suppose that $V$ is a Hilbert space over $F$, and that $T \in B(V)$.
   a) Show that $T^*T$ and $TT^*$ are both Hermitian.
   b) Show that there exist Hermitian $R, S \in B(V)$ such that $T = R + iS$. 