

# LINEAR ALGEBRA

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## Chapter 12

### COMPLEX VECTOR SPACES

#### 12.1. Complex Inner Products

Our task in this section is to define a suitable complex inner product. We begin by giving a reminder of the basics of complex vector spaces or vector spaces over  $\mathbb{C}$ .

DEFINITION. A complex vector space  $V$  is a set of objects, known as vectors, together with vector addition  $+$  and multiplication of vectors by elements of  $\mathbb{C}$ , and satisfying the following properties:

- (VA1) For every  $\mathbf{u}, \mathbf{v} \in V$ , we have  $\mathbf{u} + \mathbf{v} \in V$ .
- (VA2) For every  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , we have  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- (VA3) There exists an element  $\mathbf{0} \in V$  such that for every  $\mathbf{u} \in V$ , we have  $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ .
- (VA4) For every  $\mathbf{u} \in V$ , there exists  $-\mathbf{u} \in V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- (VA5) For every  $\mathbf{u}, \mathbf{v} \in V$ , we have  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- (SM1) For every  $c \in \mathbb{C}$  and  $\mathbf{u} \in V$ , we have  $c\mathbf{u} \in V$ .
- (SM2) For every  $c \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , we have  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .
- (SM3) For every  $a, b \in \mathbb{C}$  and  $\mathbf{u} \in V$ , we have  $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ .
- (SM4) For every  $a, b \in \mathbb{C}$  and  $\mathbf{u} \in V$ , we have  $(ab)\mathbf{u} = a(b\mathbf{u})$ .
- (SM5) For every  $\mathbf{u} \in V$ , we have  $1\mathbf{u} = \mathbf{u}$ .

REMARK. Subspaces of complex vector spaces can be defined in a similar way as for real vector spaces.

An example of a complex vector space is the euclidean space  $\mathbb{C}^n$  consisting of all vectors of the form  $\mathbf{u} = (u_1, \dots, u_n)$ , where  $u_1, \dots, u_n \in \mathbb{C}$ . We shall first generalize the concept of dot product, norm and distance, first developed for  $\mathbb{R}^n$  in Chapter 9.

DEFINITION. Suppose that  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  are vectors in  $\mathbb{C}^n$ . The complex euclidean inner product of  $\mathbf{u}$  and  $\mathbf{v}$  is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 \bar{v}_1 + \dots + u_n \bar{v}_n,$$

the complex euclidean norm of  $\mathbf{u}$  is defined by

$$\|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{1/2} = (|u_1|^2 + \dots + |u_n|^2)^{1/2},$$

and the complex euclidean distance between  $\mathbf{u}$  and  $\mathbf{v}$  is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = (|u_1 - v_1|^2 + \dots + |u_n - v_n|^2)^{1/2}.$$

Corresponding to Proposition 9A, we have the following result.

**PROPOSITION 12A.** *Suppose that  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$  and  $c \in \mathbb{C}$ . Then*

- (a)  $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$ ;
- (b)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$ ;
- (c)  $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$ ; and
- (d)  $\mathbf{u} \cdot \mathbf{u} \geq 0$ , and  $\mathbf{u} \cdot \mathbf{u} = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

The following definition is motivated by Proposition 12A.

**DEFINITION.** Suppose that  $V$  is a complex vector space. By a complex inner product on  $V$ , we mean a function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$  which satisfies the following conditions:

- (IP1) For every  $\mathbf{u}, \mathbf{v} \in V$ , we have  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ .
- (IP2) For every  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , we have  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ .
- (IP3) For every  $\mathbf{u}, \mathbf{v} \in V$  and  $c \in \mathbb{C}$ , we have  $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$ .
- (IP4) For every  $\mathbf{u} \in V$ , we have  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ , and  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

**DEFINITION.** A complex vector space with an inner product is called a complex inner product space or a unitary space.

**DEFINITION.** Suppose that  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in a complex inner product space  $V$ . Then the norm of  $\mathbf{u}$  is defined by

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2},$$

and the distance between  $\mathbf{u}$  and  $\mathbf{v}$  is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Using this inner product, we can discuss orthogonality, orthogonal and orthonormal bases, the Gram-Schmidt orthogonalization process, as well as orthogonal projections, in a similar way as for real inner product spaces. In particular, the results in Sections 9.4 and 9.5 can be generalized to the case of complex inner product spaces.

## 12.2. Unitary Matrices

For matrices with real entries, orthogonal matrices and symmetric matrices play an important role in the orthogonal diagonalization problem. For matrices with complex entries, the analogous roles are played by unitary matrices and hermitian matrices respectively.

**DEFINITION.** Suppose that  $A$  is a matrix with complex entries. Suppose further that the matrix  $\overline{A}$  is obtained from the matrix  $A$  by replacing each entry of  $A$  by its complex conjugate. Then the matrix

$$A^* = \overline{A}^t$$

is called the conjugate transpose of the matrix  $A$ .

**PROPOSITION 12B.** *Suppose that  $A$  and  $B$  are matrices with complex entries, and that  $c \in \mathbb{C}$ . Then*

- (a)  $(A^*)^* = A$ ;
- (b)  $(A + B)^* = A^* + B^*$ ;
- (c)  $(cA)^* = \bar{c}A^*$ ; and
- (d)  $(AB)^* = B^*A^*$ .

**DEFINITION.** A square matrix  $A$  with complex entries and satisfying the condition  $A^{-1} = A^*$  is said to be a unitary matrix.

Corresponding to Proposition 10B, we have the following result.

**PROPOSITION 12C.** *Suppose that  $A$  is an  $n \times n$  matrix with complex entries. Then*

- (a)  $A$  is unitary if and only if the row vectors of  $A$  form an orthonormal basis of  $\mathbb{C}^n$  under the complex euclidean inner product; and
- (b)  $A$  is unitary if and only if the column vectors of  $A$  form an orthonormal basis of  $\mathbb{C}^n$  under the complex euclidean inner product.

### 12.3. Unitary Diagonalization

Corresponding to the orthogonal diagonalization problem in Section 10.3, we now discuss the following unitary diagonalization problem.

**DEFINITION.** A square matrix  $A$  with complex entries is said to be unitarily diagonalizable if there exists a unitary matrix  $P$  with complex entries such that  $P^{-1}AP = P^*AP$  is a diagonal matrix with complex entries.

First of all, we would like to determine which matrices are unitarily diagonalizable. For those that are, we then need to discuss how we may find a unitary matrix  $P$  to carry out the diagonalization. As before, we study the question of eigenvalues and eigenvectors of a given matrix; these are defined as for the real case without any change.

In Section 10.3, we have indicated that a square matrix with real entries is orthogonally diagonalizable if and only if it is symmetric. The most natural extension to the complex case is the following.

**DEFINITION.** A square matrix  $A$  with complex entries is said to be hermitian if  $A = A^*$ .

Unfortunately, it is not true that a square matrix with complex entries is unitarily diagonalizable if and only if it is hermitian. While it is true that every hermitian matrix is unitarily diagonalizable, there are unitarily diagonalizable matrices that are not hermitian. The explanation is provided by the following.

**DEFINITION.** A square matrix  $A$  with complex entries is said to be normal if  $AA^* = A^*A$ .

**REMARK.** Note that every hermitian matrix is normal and every unitary matrix is normal.

Corresponding to Propositions 10E and 10G, we have the following results.

**PROPOSITION 12D.** *Suppose that  $A$  is an  $n \times n$  matrix with complex entries. Then it is unitarily diagonalizable if and only if it is normal.*

**PROPOSITION 12E.** *Suppose that  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are eigenvectors of a normal matrix  $A$  with complex entries, corresponding to distinct eigenvalues  $\lambda_1$  and  $\lambda_2$  respectively. Then  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ . In other words, eigenvectors of a normal matrix corresponding to distinct eigenvalues are orthogonal.*

We can now follow the procedure below.

**UNITARY DIAGONALIZATION PROCESS.** Suppose that  $A$  is a normal  $n \times n$  matrix with complex entries.

- (1) Determine the  $n$  complex roots  $\lambda_1, \dots, \lambda_n$  of the characteristic polynomial  $\det(A - \lambda I)$ , and find  $n$  linearly independent eigenvectors  $\mathbf{u}_1, \dots, \mathbf{u}_n$  of  $A$  corresponding to these eigenvalues as in the Diagonalization process.
- (2) Apply the Gram-Schmidt orthogonalization process to the eigenvectors  $\mathbf{u}_1, \dots, \mathbf{u}_n$  to obtain orthogonal eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $A$ , noting that eigenvectors corresponding to distinct eigenvalues are already orthogonal.
- (3) Normalize the orthogonal eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  to obtain orthonormal eigenvectors  $\mathbf{w}_1, \dots, \mathbf{w}_n$  of  $A$ . These form an orthonormal basis of  $\mathbb{C}^n$ . Furthermore, write

$$P = (\mathbf{w}_1 \quad \dots \quad \mathbf{w}_n) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  are the eigenvalues of  $A$  and where  $\mathbf{w}_1, \dots, \mathbf{w}_n \in \mathbb{C}^n$  are respectively their orthogonalized and normalized eigenvectors. Then  $P^*AP = D$ .

We conclude this chapter by discussing the following important result which implies Proposition 10F, that all the eigenvalues of a symmetric real matrix are real.

**PROPOSITION 12F.** Suppose that  $A$  is a hermitian matrix. Then all the eigenvalues of  $A$  are real.

**SKETCH OF PROOF.** Suppose that  $A$  is a hermitian matrix. Suppose further that  $\lambda$  is an eigenvalue of  $A$ , with corresponding eigenvector  $\mathbf{v}$ . Then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Multiplying on the left by the conjugate transpose  $\mathbf{v}^*$  of  $\mathbf{v}$ , we obtain

$$\mathbf{v}^*A\mathbf{v} = \mathbf{v}^*\lambda\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v}.$$

To show that  $\lambda$  is real, it suffices to show that the  $1 \times 1$  matrices  $\mathbf{v}^*A\mathbf{v}$  and  $\mathbf{v}^*\mathbf{v}$  both have real entries. Now

$$(\mathbf{v}^*A\mathbf{v})^* = \mathbf{v}^*A^*(\mathbf{v}^*)^* = \mathbf{v}^*A\mathbf{v}$$

and

$$(\mathbf{v}^*\mathbf{v})^* = \mathbf{v}^*(\mathbf{v}^*)^* = \mathbf{v}^*\mathbf{v}.$$

It follows that both  $\mathbf{v}^*A\mathbf{v}$  and  $\mathbf{v}^*\mathbf{v}$  are hermitian. It is easy to prove that hermitian matrices must have real entries on the main diagonal. Since  $\mathbf{v}^*A\mathbf{v}$  and  $\mathbf{v}^*\mathbf{v}$  are  $1 \times 1$ , it follows that they are real.  $\circ$

## PROBLEMS FOR CHAPTER 12

1. Consider the set  $V$  of all matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix},$$

where  $z \in \mathbb{C}$ , with matrix addition and scalar multiplication. Determine whether  $V$  forms a complex vector space.

2. Is  $\mathbb{R}^n$  a subspace of  $\mathbb{C}^n$ ? Justify your assertion.
3. Prove Proposition 12A.
4. Suppose that  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  are elements of a complex inner product space, and that  $c \in \mathbb{C}$ .
- Show that  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ .
  - Show that  $\langle \mathbf{u}, c\mathbf{v} \rangle = \bar{c}\langle \mathbf{u}, \mathbf{v} \rangle$ .
5. Let  $V$  be the vector space of all continuous functions  $f : [0, 1] \rightarrow \mathbb{C}$ . Show that

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} dx$$

defines a complex inner product on  $V$ .

6. Suppose that  $\mathbf{u}, \mathbf{v}$  are elements of a complex inner product space, and that  $c \in \mathbb{C}$ .
- Show that  $\langle \mathbf{u} - c\mathbf{v}, \mathbf{u} - c\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - \bar{c}\langle \mathbf{u}, \mathbf{v} \rangle - c\langle \mathbf{u}, \mathbf{v} \rangle + c\bar{c}\langle \mathbf{v}, \mathbf{v} \rangle$ .
  - Deduce that  $\langle \mathbf{u}, \mathbf{u} \rangle - \bar{c}\langle \mathbf{u}, \mathbf{v} \rangle - c\langle \mathbf{u}, \mathbf{v} \rangle + c\bar{c}\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ .
  - Prove the Cauchy-Schwarz inequality, that  $|\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq \langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle$ .
7. Generalize the results in Sections 9.4 and 9.5 to the case of complex inner product spaces. Try to prove as many results as possible.
8. Prove Proposition 12B.
9. Prove Proposition 12C.
10. Prove that the diagonal entries on every hermitian matrix are all real.
11. Suppose that  $A$  is a square matrix with complex entries.
- Prove that  $\det(\overline{A}) = \overline{\det A}$ .
  - Deduce that  $\det(A^*) = \overline{\det A}$ .
  - Prove that if  $A$  is hermitian, then  $\det A$  is real.
  - Prove that if  $A$  is unitary, then  $|\det A| = 1$ .

12. Apply the Unitary diagonalization process to each of the following matrices:

a)  $A = \begin{pmatrix} 4 & 1 - i \\ 1 + i & 5 \end{pmatrix}$

b)  $A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}$

c)  $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & -1 + i \\ 0 & -1 - i & 0 \end{pmatrix}$

13. Suppose that  $\lambda_1$  and  $\lambda_2$  are distinct eigenvalues of a hermitian matrix  $A$ , with eigenvectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  respectively.
- Show that  $\mathbf{u}_1^* A \mathbf{u}_2 = \lambda_1 \mathbf{u}_1^* \mathbf{u}_2$  and  $\mathbf{u}_1^* A \mathbf{u}_2 = \lambda_2 \mathbf{u}_1^* \mathbf{u}_2$ .
  - Complete the proof of Proposition 12E.
14. Suppose that  $A$  is a square matrix with complex entries, and that  $A^* = -A$ .
- Show that  $iA$  is a hermitian matrix.
  - Show that  $A$  is unitarily diagonalizable but has purely imaginary eigenvalues.