

# LINEAR ALGEBRA

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## Chapter 11

### APPLICATIONS OF REAL INNER PRODUCT SPACES

#### 11.1. Least Squares Approximation

Given a continuous function  $f : [a, b] \rightarrow \mathbb{R}$ , we wish to approximate  $f$  by a polynomial  $g : [a, b] \rightarrow \mathbb{R}$  of degree at most  $k$ , such that the error

$$\int_a^b |f(x) - g(x)|^2 dx$$

is minimized. The purpose of this section is to study this problem using the theory of real inner product spaces. Our argument is underpinned by the following simple result in the theory.

**PROPOSITION 11A.** *Suppose that  $V$  is a real inner product space, and that  $W$  is a finite-dimensional subspace of  $V$ . Given any  $\mathbf{u} \in V$ , the inequality*

$$\|\mathbf{u} - \text{proj}_W \mathbf{u}\| \leq \|\mathbf{u} - \mathbf{w}\|$$

*holds for every  $\mathbf{w} \in W$ .*

In other words, the distance from  $\mathbf{u}$  to any  $\mathbf{w} \in W$  is minimized by the choice  $\mathbf{w} = \text{proj}_W \mathbf{u}$ , the orthogonal projection of  $\mathbf{u}$  on the subspace  $W$ . Alternatively,  $\text{proj}_W \mathbf{u}$  can be thought of as the vector in  $W$  closest to  $\mathbf{u}$ .

PROOF OF PROPOSITION 11A. Note that

$$\mathbf{u} - \text{proj}_W \mathbf{u} \in W^\perp \quad \text{and} \quad \text{proj}_W \mathbf{u} - \mathbf{w} \in W.$$

It follows from Pythagoras's theorem that

$$\|\mathbf{u} - \mathbf{w}\|^2 = \|(\mathbf{u} - \text{proj}_W \mathbf{u}) + (\text{proj}_W \mathbf{u} - \mathbf{w})\|^2 = \|\mathbf{u} - \text{proj}_W \mathbf{u}\|^2 + \|\text{proj}_W \mathbf{u} - \mathbf{w}\|^2,$$

so that

$$\|\mathbf{u} - \mathbf{w}\|^2 - \|\mathbf{u} - \text{proj}_W \mathbf{u}\|^2 = \|\text{proj}_W \mathbf{u} - \mathbf{w}\|^2 \geq 0.$$

The result follows immediately.  $\circ$

Let  $V$  denote the vector space  $C[a, b]$  of all continuous real valued functions on the closed interval  $[a, b]$ , with inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, dx.$$

Then

$$\int_a^b |f(x) - g(x)|^2 \, dx = \langle f - g, f - g \rangle = \|f - g\|^2.$$

It follows that the least squares approximation problem is reduced to one of finding a suitable polynomial  $g$  to minimize the norm  $\|f - g\|$ .

Now let  $W = P_k[a, b]$  be the collection of all polynomials  $g : [a, b] \rightarrow \mathbb{R}$  with real coefficients and of degree at most  $k$ . Note that  $W$  is essentially  $P_k$ , although the variable is restricted to the closed interval  $[a, b]$ . It is easy to show that  $W$  is a subspace of  $V$ . In view of Proposition 11A, we conclude that

$$g = \text{proj}_W f$$

gives the best least squares approximation among polynomials in  $W = P_k[a, b]$ . This subspace is of dimension  $k + 1$ . Suppose that  $\{v_0, v_1, \dots, v_k\}$  is an orthogonal basis of  $W = P_k[a, b]$ . Then by Proposition 9L, we have

$$g = \frac{\langle f, v_0 \rangle}{\|v_0\|^2} v_0 + \frac{\langle f, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle f, v_k \rangle}{\|v_k\|^2} v_k.$$

**EXAMPLE 11.1.1.** Consider the function  $f(x) = x^2$  in the interval  $[0, 2]$ . Suppose that we wish to find a least squares approximation by a polynomial of degree at most 1. In this case, we can take  $V = C[0, 2]$ , with inner product

$$\langle f, g \rangle = \int_0^2 f(x)g(x) \, dx,$$

and  $W = P_1[0, 2]$ , with basis  $\{1, x\}$ . We now apply the Gram-Schmidt orthogonalization process to this basis to obtain an orthogonal basis  $\{1, x - 1\}$  of  $W$ , and take

$$g = \frac{\langle x^2, 1 \rangle}{\|1\|^2} 1 + \frac{\langle x^2, x - 1 \rangle}{\|x - 1\|^2} (x - 1).$$

Now

$$\langle x^2, 1 \rangle = \int_0^2 x^2 \, dx = \frac{8}{3} \quad \text{and} \quad \|1\|^2 = \langle 1, 1 \rangle = \int_0^2 dx = 2,$$

while

$$\langle x^2, x - 1 \rangle = \int_0^2 x^2(x - 1) dx = \frac{4}{3} \quad \text{and} \quad \|x - 1\|^2 = \langle x - 1, x - 1 \rangle = \int_0^2 (x - 1)^2 dx = \frac{2}{3}.$$

It follows that

$$g = \frac{4}{3} + 2(x - 1) = 2x - \frac{2}{3}.$$

EXAMPLE 11.1.2. Consider the function  $f(x) = e^x$  in the interval  $[0, 1]$ . Suppose that we wish to find a least squares approximation by a polynomial of degree at most 1. In this case, we can take  $V = C[0, 1]$ , with inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx,$$

and  $W = P_1[0, 1]$ , with basis  $\{1, x\}$ . We now apply the Gram-Schmidt orthogonalization process to this basis to obtain an orthogonal basis  $\{1, x - 1/2\}$  of  $W$ , and take

$$g = \frac{\langle e^x, 1 \rangle}{\|1\|^2} 1 + \frac{\langle e^x, x - 1/2 \rangle}{\|x - 1/2\|^2} \left(x - \frac{1}{2}\right).$$

Now

$$\langle e^x, 1 \rangle = \int_0^1 e^x dx = e - 1 \quad \text{and} \quad \langle e^x, x \rangle = \int_0^1 e^x x dx = 1,$$

so that

$$\left\langle e^x, x - \frac{1}{2} \right\rangle = \langle e^x, x \rangle - \frac{1}{2} \langle e^x, 1 \rangle = \frac{3}{2} - \frac{e}{2}.$$

Also

$$\|1\|^2 = \langle 1, 1 \rangle = \int_0^1 dx = 1 \quad \text{and} \quad \left\|x - \frac{1}{2}\right\|^2 = \left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}.$$

It follows that

$$g = (e - 1) + (18 - 6e) \left(x - \frac{1}{2}\right) = (18 - 6e)x + (4e - 10).$$

REMARK. From the proof of Proposition 11A, it is clear that  $\|\mathbf{u} - \mathbf{w}\|$  is minimized by the unique choice  $\mathbf{w} = \text{proj}_W \mathbf{u}$ . It follows that the least squares approximation problem posed here has a unique solution.

### 11.2. Quadratic Forms

A real quadratic form in  $n$  variables  $x_1, \dots, x_n$  is an expression of the form

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ i \leq j}}^n c_{ij} x_i x_j, \tag{1}$$

where  $c_{ij} \in \mathbb{R}$  for every  $i, j = 1, \dots, n$  satisfying  $i \leq j$ .

EXAMPLE 11.2.1. The expression  $5x_1^2 + 6x_1x_2 + 7x_2^2$  is a quadratic form in two variables  $x_1$  and  $x_2$ . It can be written in the form

$$5x_1^2 + 6x_1x_2 + 7x_2^2 = (x_1 \ x_2) \begin{pmatrix} 5 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

EXAMPLE 11.2.2. The expression  $4x_1^2 + 5x_2^2 + 3x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3$  is a quadratic form in three variables  $x_1, x_2$  and  $x_3$ . It can be written in the form

$$4x_1^2 + 5x_2^2 + 3x_3^2 + 2x_1x_2 + 4x_1x_3 + 6x_2x_3 = (x_1 \ x_2 \ x_3) \begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Note that in both examples, the quadratic form can be described in terms of a real symmetric matrix. In fact, this is always possible. To see this, note that given any quadratic form (1), we can write, for every  $i, j = 1, \dots, n$ ,

$$a_{ij} = \begin{cases} c_{ij} & \text{if } i = j, \\ \frac{1}{2}c_{ij} & \text{if } i < j, \\ \frac{1}{2}c_{ji} & \text{if } i > j. \end{cases} \quad (2)$$

Then

$$\sum_{\substack{i=1 \\ i \leq j}}^n \sum_{j=1}^n c_{ij}x_i x_j = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j = (x_1 \ \dots \ x_n) \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is clearly symmetric, in view of (2).

We are interested in the case when  $x_1, \dots, x_n$  take real values. In this case, we can write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

It follows that a quadratic form can be written as

$$\mathbf{x}^t A \mathbf{x},$$

where  $A$  is an  $n \times n$  real symmetric matrix and  $\mathbf{x}$  takes values in  $\mathbb{R}^n$ .

Many problems in mathematics can be studied using quadratic forms. Here we shall restrict our attention to two fundamental problems which are in fact related. The first is the question of what conditions the matrix  $A$  must satisfy in order that the inequality

$$\mathbf{x}^t A \mathbf{x} > 0$$

holds for every non-zero  $\mathbf{x} \in \mathbb{R}^n$ . The second is the question of whether it is possible to have a change of variables of the type  $\mathbf{x} = P\mathbf{y}$ , where  $P$  is an invertible matrix, such that the quadratic form  $\mathbf{x}^t A \mathbf{x}$  can be represented in the alternative form  $\mathbf{y}^t D \mathbf{y}$ , where  $D$  is a diagonal matrix with real entries.

DEFINITION. A quadratic form  $\mathbf{x}^t A \mathbf{x}$  is said to be positive definite if  $\mathbf{x}^t A \mathbf{x} > 0$  for every non-zero  $\mathbf{x} \in \mathbb{R}^n$ . In this case, we say that the symmetric matrix  $A$  is a positive definite matrix.

To answer our first question, we shall prove the following result.

**PROPOSITION 11B.** *A quadratic form  $\mathbf{x}^t A \mathbf{x}$  is positive definite if and only if all the eigenvalues of the symmetric matrix  $A$  are positive.*

Our strategy here is to prove Proposition 11B by first studying our second question. Since the matrix  $A$  is real and symmetric, it follows from Proposition 10E that it is orthogonally diagonalizable. In other words, there exists an orthogonal matrix  $P$  and a diagonal matrix  $D$  such that  $P^t A P = D$ , and so  $A = P D P^t$ . It follows that

$$\mathbf{x}^t A \mathbf{x} = \mathbf{x}^t P D P^t \mathbf{x},$$

and so, writing

$$\mathbf{y} = P^t \mathbf{x},$$

we have

$$\mathbf{x}^t A \mathbf{x} = \mathbf{y}^t D \mathbf{y}.$$

Also, since  $P$  is an orthogonal matrix, we also have  $\mathbf{x} = P \mathbf{y}$ . This answers our second question.

Furthermore, in view of the Orthogonal diagonalization process, the diagonal entries in the matrix  $D$  can be taken to be the eigenvalues of  $A$ , so that

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  are the eigenvalues of  $A$ . Writing

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

we have

$$\mathbf{x}^t A \mathbf{x} = \mathbf{y}^t D \mathbf{y} = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2. \tag{3}$$

Note now that  $\mathbf{x} = \mathbf{0}$  if and only if  $\mathbf{y} = \mathbf{0}$ , since  $P$  is an invertible matrix. Proposition 11B now follows immediately from (3).

EXAMPLE 11.2.3. Consider the quadratic form  $2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$ . This can be written in the form  $\mathbf{x}^t A \mathbf{x}$ , where

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The matrix  $A$  has eigenvalues  $\lambda_1 = 7$  and (double root)  $\lambda_2 = \lambda_3 = 1$ ; see Example 10.3.1. Furthermore, we have  $P^t A P = D$ , where

$$P = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Writing  $\mathbf{y} = P^t \mathbf{x}$ , the quadratic form becomes  $7y_1^2 + y_2^2 + y_3^2$  which is clearly positive definite.

EXAMPLE 11.2.4. Consider the quadratic form  $5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3$ . This can be written in the form  $\mathbf{x}^t A \mathbf{x}$ , where

$$A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The matrix  $A$  has eigenvalues  $\lambda_1 = 3$ ,  $\lambda_2 = 6$  and  $\lambda_3 = 9$ ; see Example 10.3.3. Furthermore, we have  $P^t A P = D$ , where

$$P = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Writing  $\mathbf{y} = P^t \mathbf{x}$ , the quadratic form becomes  $3y_1^2 + 6y_2^2 + 9y_3^2$  which is clearly positive definite.

EXAMPLE 11.2.5. Consider the quadratic form  $x_1^2 + x_2^2 + 2x_1x_2$ . Clearly this is equal to  $(x_1 + x_2)^2$  and is therefore not positive definite. The quadratic form can be written in the form  $\mathbf{x}^t A \mathbf{x}$ , where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It follows from Proposition 11B that the eigenvalues of  $A$  are not all positive. Indeed, the matrix  $A$  has eigenvalues  $\lambda_1 = 2$  and  $\lambda_2 = 0$ , with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence we may take

$$P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing  $\mathbf{y} = P^t \mathbf{x}$ , the quadratic form becomes  $2y_1^2$  which is not positive definite.

### 11.3. Real Fourier Series

Let  $E$  denote the collection of all functions  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  which are piecewise continuous on the interval  $[-\pi, \pi]$ . This means that any  $f \in E$  has at most a finite number of points of discontinuity, at each of which  $f$  need not be defined but must have one sided limits which are finite. We further adopt the convention that any two functions  $f, g \in E$  are considered equal, denoted by  $f = g$ , if  $f(x) = g(x)$  for every  $x \in [-\pi, \pi]$  with at most a finite number of exceptions.

It is easy to check that  $E$  forms a real vector space. More precisely, let  $\lambda \in E$  denote the function  $\lambda : [-\pi, \pi] \rightarrow \mathbb{R}$ , where  $\lambda(x) = 0$  for every  $x \in [-\pi, \pi]$ . Then the following conditions hold:

- For every  $f, g \in E$ , we have  $f + g \in E$ .
- For every  $f, g, h \in E$ , we have  $f + (g + h) = (f + g) + h$ .
- For every  $f \in E$ , we have  $f + \lambda = \lambda + f = f$ .
- For every  $f \in E$ , we have  $f + (-f) = \lambda$ .
- For every  $f, g \in E$ , we have  $f + g = g + f$ .
- For every  $c \in \mathbb{R}$  and  $f \in E$ , we have  $cf \in E$ .
- For every  $c \in \mathbb{R}$  and  $f, g \in E$ , we have  $c(f + g) = cf + cg$ .
- For every  $a, b \in \mathbb{R}$  and  $f \in E$ , we have  $(a + b)f = af + bf$ .
- For every  $a, b \in \mathbb{R}$  and  $f \in E$ , we have  $(ab)f = a(bf)$ .
- For every  $f \in E$ , we have  $1f = f$ .

We now give this vector space  $E$  more structure by introducing an inner product. For every  $f, g \in E$ , write

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) \, dx.$$

The integral exists since the function  $f(x)g(x)$  is clearly piecewise continuous on  $[-\pi, \pi]$ . It is easy to check that the following conditions hold:

- For every  $f, g \in E$ , we have  $\langle f, g \rangle = \langle g, f \rangle$ .
- For every  $f, g, h \in E$ , we have  $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$ .
- For every  $f, g \in E$  and  $c \in \mathbb{R}$ , we have  $c\langle f, g \rangle = \langle cf, g \rangle$ .
- For every  $f \in E$ , we have  $\langle f, f \rangle \geq 0$ , and  $\langle f, f \rangle = 0$  if and only if  $f = \lambda$ .

Hence  $E$  is a real inner product space.

The difficulty here is that the inner product space  $E$  is not finite-dimensional. It is not straightforward to show that the set

$$\left\{ \frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \dots \right\} \tag{4}$$

in  $E$  forms an orthonormal “basis” for  $E$ . The difficulty is to show that the set spans  $E$ .

REMARK. It is easy to check that the elements in (4) form an orthonormal “system”. For every  $k, m \in \mathbb{N}$ , we have

$$\begin{aligned} \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \, dx = 1; \\ \left\langle \frac{1}{\sqrt{2}}, \sin kx \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin kx \, dx = 0; \\ \left\langle \frac{1}{\sqrt{2}}, \cos kx \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos kx \, dx = 0; \end{aligned}$$

as well as

$$\begin{aligned} \langle \sin kx, \sin mx \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \sin mx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos(k-m)x - \cos(k+m)x) \, dx = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m; \end{cases} \\ \langle \cos kx, \cos mx \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx \cos mx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos(k-m)x + \cos(k+m)x) \, dx = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m; \end{cases} \end{aligned}$$

and

$$\langle \sin kx, \cos mx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx \cos mx \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\sin(k-m)x + \sin(k+m)x) \, dx = 0.$$

Let us assume that we have established that the set (4) forms an orthonormal basis for  $E$ . Then a natural extension of Proposition 9H gives rise to the following: Every function  $f \in E$  can be written uniquely in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{5}$$

known usually as the (trigonometric) Fourier series of the function  $f$ , with Fourier coefficients

$$\frac{a_0}{\sqrt{2}} = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,$$

and, for every  $n \in \mathbb{N}$ ,

$$a_n = \langle f, \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad \text{and} \quad b_n = \langle f, \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Note that the constant term in the Fourier series (5) is given by

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{a_0}{2}.$$

EXAMPLE 11.3.1. Consider the function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$ , given by  $f(x) = x$  for every  $x \in [-\pi, \pi]$ . For every  $n \in \mathbb{N} \cup \{0\}$ , we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx = 0,$$

since the integrand is an odd function. On the other hand, for every  $n \in \mathbb{N}$ , we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx,$$

since the integrand is an even function. On integrating by parts, we have

$$b_n = \frac{2}{\pi} \left( - \left[ \frac{x \cos nx}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \, dx \right) = \frac{2}{\pi} \left( - \left[ \frac{x \cos nx}{n} \right]_0^{\pi} + \left[ \frac{\sin nx}{n^2} \right]_0^{\pi} \right) = \frac{2(-1)^{n+1}}{n}.$$

We therefore have the (trigonometric) Fourier series

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

Note that the function  $f$  is odd, and this plays a crucial role in eschewing the Fourier coefficients  $a_n$  corresponding to the even part of the Fourier series.

EXAMPLE 11.3.2. Consider the function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$ , given by  $f(x) = |x|$  for every  $x \in [-\pi, \pi]$ . For every  $n \in \mathbb{N} \cup \{0\}$ , we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx,$$

since the integrand is an even function. Clearly

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi.$$

Furthermore, for every  $n \in \mathbb{N}$ , on integrating by parts, we have

$$a_n = \frac{2}{\pi} \left( \left[ \frac{x \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} \, dx \right) = \frac{2}{\pi} \left( \left[ \frac{x \sin nx}{n} \right]_0^{\pi} + \left[ \frac{\cos nx}{n^2} \right]_0^{\pi} \right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, for every  $n \in \mathbb{N}$ , we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx \, dx = 0,$$



since the integrand is an odd function. We therefore have the (trigonometric) Fourier series

$$\frac{\pi}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n^2} \cos nx = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \cos(2k-1)x.$$

Note that the function  $f$  is even, and this plays a crucial role in eschewing the Fourier coefficients  $b_n$  corresponding to the odd part of the Fourier series.

EXAMPLE 11.3.3. Consider the function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$ , given for every  $x \in [-\pi, \pi]$  by

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & \text{if } 0 < x \leq \pi, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } -\pi \leq x < 0. \end{cases}$$

For every  $n \in \mathbb{N} \cup \{0\}$ , we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) \cos nx \, dx = 0,$$

since the integrand is an odd function. On the other hand, for every  $n \in \mathbb{N}$ , we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sgn}(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx,$$

since the integrand is an even function. It is easy to see that

$$b_n = -\frac{2}{\pi} \left[ \frac{\cos nx}{n} \right]_0^{\pi} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{\pi n} & \text{if } n \text{ is odd.} \end{cases}$$

We therefore have the (trigonometric) Fourier series

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin nx = \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)} \sin(2k-1)x.$$

EXAMPLE 11.3.4. Consider the function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$ , given by  $f(x) = x^2$  for every  $x \in [-\pi, \pi]$ . For every  $n \in \mathbb{N} \cup \{0\}$ , we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx,$$

since the integrand is an even function. Clearly

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}.$$

Furthermore, for every  $n \in \mathbb{N}$ , on integrating by parts, we have

$$\begin{aligned} a_n &= \frac{2}{\pi} \left( \left[ \frac{x^2 \sin nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{2x \sin nx}{n} \, dx \right) = \frac{2}{\pi} \left( \left[ \frac{x^2 \sin nx}{n} \right]_0^{\pi} + \left[ \frac{2x \cos nx}{n^2} \right]_0^{\pi} - \int_0^{\pi} \frac{2 \cos nx}{n^2} \, dx \right) \\ &= \frac{2}{\pi} \left( \left[ \frac{x^2 \sin nx}{n} \right]_0^{\pi} + \left[ \frac{2x \cos nx}{n^2} \right]_0^{\pi} - \left[ \frac{2 \sin nx}{n^3} \right]_0^{\pi} \right) = \frac{4(-1)^n}{n^2}. \end{aligned}$$

On the other hand, for every  $n \in \mathbb{N}$ , we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0,$$

since the integrand is an odd function. We therefore have the (trigonometric) Fourier series

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx.$$

## PROBLEMS FOR CHAPTER 11

1. Consider the function  $f : [-1, 1] \rightarrow \mathbb{R} : x \mapsto x^3$ . We wish to find a polynomial  $g(x) = ax + b$  which minimizes the error

$$\int_{-1}^1 |f(x) - g(x)|^2 dx.$$

Follow the steps below to find this polynomial  $g$ :

- Consider the real vector space  $C[-1, 1]$ . Write down a suitable real inner product on  $C[-1, 1]$  for this problem, explaining carefully the steps that you take.
  - Consider now the subspace  $P_1[-1, 1]$  of all polynomials of degree at most 1. Describe the polynomial  $g$  in terms of  $f$  and orthogonal projection with respect to the inner product in part (a). Give a brief explanation for your choice.
  - Write down a basis of  $P_1[-1, 1]$ .
  - Apply the Gram-Schmidt process to your basis in part (c) to obtain an orthogonal basis of  $P_1[-1, 1]$ .
  - Describe your polynomial in part (b) as a linear combination of the elements of your basis in part (d), and find the precise values of the coefficients.
2. For each of the following functions, find the best least squares approximation by linear polynomials of the form  $ax + b$ , where  $a, b \in \mathbb{R}$ :
- $f : [0, \pi/2] \rightarrow \mathbb{R} : x \mapsto \sin x$
  - $f : [0, 1] \rightarrow \mathbb{R} : x \mapsto x^3$
  - $f : [0, 2] \rightarrow \mathbb{R} : x \mapsto e^x$

3. Consider the quadratic form  $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$  in three variables  $x_1, x_2, x_3$ .
- Write the quadratic form in the form  $\mathbf{x}^t A \mathbf{x}$ , where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and where  $A$  is a symmetric matrix with real entries.

- Apply the Orthogonal diagonalization process to the matrix  $A$ .
- Find a transformation of the type  $\mathbf{x} = P\mathbf{y}$ , where  $P$  is an invertible matrix, so that the quadratic form can be written as  $\mathbf{y}^t D \mathbf{y}$ , where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and where  $D$  is a diagonal matrix with real entries. You should give the matrices  $P$  and  $D$  explicitly.

- Is the quadratic form positive definite? Justify your assertion both in terms of the eigenvalues of  $A$  and in terms of your solution to part (c).
4. For each of the following quadratic forms in three variables, write it in the form  $\mathbf{x}^t A \mathbf{x}$ , find a substitution  $\mathbf{x} = P\mathbf{y}$  so that it can be written as a diagonal form in the variables  $y_1, y_2, y_3$ , and determine whether the quadratic form is positive definite:
- $x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_2 + 4x_1x_3 + 4x_2x_3$
  - $3x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_3$
  - $3x_1^2 + 5x_2^2 + 4x_3^2 + 4x_1x_3 - 4x_2x_3$
  - $5x_1^2 + 2x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 - 4x_2x_3$
  - $x_1^2 - 5x_2^2 - x_3^2 + 4x_1x_2 + 6x_2x_3$

5. Determine which of the following matrices are positive definite:

a) 
$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

b) 
$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

c) 
$$\begin{pmatrix} 6 & 1 & 7 \\ 1 & 1 & 2 \\ 7 & 2 & 9 \end{pmatrix}$$

d) 
$$\begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

e) 
$$\begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

f) 
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

6. Find the trigonometric Fourier series for each of the following functions  $f : [-\pi, \pi] \rightarrow \mathbb{C}$ :

a)  $f(x) = x|x|$  for every  $x \in [-\pi, \pi]$

b)  $f(x) = |\sin x|$  for every  $x \in [-\pi, \pi]$

c)  $f(x) = |\cos x|$  for every  $x \in [-\pi, \pi]$

d)  $f(x) = 0$  for every  $x \in [-\pi, 0]$  and  $f(x) = x$  for every  $x \in (0, \pi]$

e)  $f(x) = \sin x$  for every  $x \in [-\pi, 0]$  and  $f(x) = \cos x$  for every  $x \in (0, \pi]$

f)  $f(x) = \cos x$  for every  $x \in [-\pi, 0]$  and  $f(x) = \sin x$  for every  $x \in (0, \pi]$

g)  $f(x) = \cos(x/2)$  for every  $x \in [-\pi, \pi]$

h)  $f(x) = \sin(x/2)$  for every  $x \in [-\pi, \pi]$