LINEAR ALGEBRA

W W L CHEN

Chapter 1

LINEAR EQUATIONS

1.1. Introduction

EXAMPLE 1.1.1. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ 6x + 4y &= 5. \end{aligned}$$

It is easy to see that the two lines are parallel and do not intersect, so that this system of two linear equations has no solution.

EXAMPLE 1.1.2. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ x + y &= 2. \end{aligned}$$

It is easy to see that the two lines are not parallel and intersect at the point $(1, 1)$, so that this system of two linear equations has exactly one solution.

EXAMPLE 1.1.3. Try to draw the two lines

$$\begin{aligned} 3x + 2y &= 5, \\ 6x + 4y &= 10. \end{aligned}$$

It is easy to see that the two lines overlap completely, so that this system of two linear equations has infinitely many solutions.

In these three examples, we have shown that a system of two linear equations on the plane $\mathbb{R}^2$ may have no solution, one solution or infinitely many solutions. A natural question to ask is whether there can be any other conclusion. Well, we can see geometrically that two lines cannot intersect at more than one point without overlapping completely. Hence there can be no other conclusion.

In general, we shall study a system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{1}$$

with n variables $x_1, x_2, \ldots, x_n$. Here we may not be so lucky as to be able to see geometrically what is going on. We therefore need to study the problem from a more algebraic viewpoint. In this chapter, we shall confine ourselves to the simpler aspects of the problem. In Chapter 6, we shall study the problem again from the viewpoint of vector spaces.

If we omit reference to the variables, then system (1) can be represented by the array

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array} \right) \tag{2}$$

of all the coefficients. This is known as the augmented matrix of the system. Here the first row of the array represents the first linear equation, and so on.

We also write $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

represent the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

represents the variables.

EXAMPLE 1.1.4. The array

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array} \right) \tag{3}$$

represents the system of 3 linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{4}$$

with 5 variables x_1, x_2, x_3, x_4, x_5. We can also write

$$\begin{pmatrix} 1 & 3 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 2 & 4 & 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

1.2. Elementary Row Operations

Let us continue with Example 1.1.4.

EXAMPLE 1.2.1. Consider the array (3). Let us interchange the first and second rows to obtain

$$\left(\begin{array}{ccccc|c} 0 & 1 & 1 & 2 & 1 & 4 \\ 1 & 3 & 1 & 5 & 1 & 5 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array} \right).$$

Then this represents the system of equations

$$\begin{aligned} x_2 + x_3 + 2x_4 + x_5 &= 4, \\ x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{5}$$

essentially the same as the system (4), the only difference being that the first and second equations have been interchanged. Any solution of the system (4) is a solution of the system (5), and vice versa.

EXAMPLE 1.2.2. Consider the array (3). Let us add 2 times the second row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 5 & 3 & 9 & 3 & 13 \\ 0 & 1 & 1 & 2 & 1 & 4 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array} \right).$$

Then this represents the system of equations

$$\begin{aligned} x_1 + 5x_2 + 3x_3 + 9x_4 + 3x_5 &= 13, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{6}$$

essentially the same as the system (4), the only difference being that we have added 2 times the second equation to the first equation. Any solution of the system (4) is a solution of the system (6), and vice versa.

EXAMPLE 1.2.3. Consider the array (3). Let us multiply the second row by 2 to obtain

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 5 & 1 & 5 \\ 0 & 2 & 2 & 4 & 2 & 8 \\ 2 & 4 & 0 & 7 & 1 & 3 \end{array} \right).$$

Then this represents the system of equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ 2x_2 + 2x_3 + 4x_4 + 2x_5 &= 8, \\ 2x_1 + 4x_2 \qquad + 7x_4 + x_5 &= 3, \end{aligned} \tag{7}$$

essentially the same as the system (4), the only difference being that the second equation has been multiplied through by 2. Any solution of the system (4) is a solution of the system (7), and vice versa.

In the general situation, it is not difficult to see the following.

PROPOSITION 1A. (ELEMENTARY ROW OPERATIONS) *Consider the array* (2) *corresponding to the system* (1)*.*
(a) Interchanging the i-th and j-th rows of (2) *corresponds to interchanging the i-th and j-th equations in* (1)*.*
(b) Adding a multiple of the i-th row of (2) *to the j-th row corresponds to adding the same multiple of the i-th equation in* (1) *to the j-th equation.*
(c) Multiplying the i-th row of (2) *by a non-zero constant corresponds to multiplying the i-th equation in* (1) *by the same non-zero constant.*
In all three cases, the collection of solutions to the system (1) *remains unchanged.*

Let us investigate how we may use elementary row operations to help us solve a system of linear equations. As a first step, let us continue again with Example 1.1.4.

Example 1.2.4. Consider again the system of linear equations

$$\begin{aligned} x_1+3x_2+x_3+5x_4+x_5&=5,\\ x_2+x_3+2x_4+x_5&=4,\\ 2x_1+4x_2\qquad+7x_4+x_5&=3, \end{aligned} \tag{8}$$

represented by the array

$$\left(\begin{array}{ccccc|c} 1&3&1&5&1&5\\ 0&1&1&2&1&4\\ 2&4&0&7&1&3 \end{array}\right). \tag{9}$$

Let us now perform elementary row operations on the array (9). At this point, do not worry if you do not understand why we are taking the following steps. Adding -2 times the first row of (9) to the third row, we obtain

$$\left(\begin{array}{ccccc|c} 1&3&1&5&1&5\\ 0&1&1&2&1&4\\ 0&-2&-2&-3&-1&-7 \end{array}\right).$$

From here, we add 2 times the second row to the third row to obtain

$$\left(\begin{array}{ccccc|c} 1&3&1&5&1&5\\ 0&1&1&2&1&4\\ 0&0&0&1&1&1 \end{array}\right). \tag{10}$$

Next, we add -3 times the second row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1&0&-2&-1&-2&-7\\ 0&1&1&2&1&4\\ 0&0&0&1&1&1 \end{array}\right).$$

Next, we add the third row to the first row to obtain

$$\left(\begin{array}{ccccc|c} 1&0&-2&0&-1&-6\\ 0&1&1&2&1&4\\ 0&0&0&1&1&1 \end{array}\right).$$

Finally, we add -2 times the third to row to the second row to obtain

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & -1 & -6 \\ 0 & 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \tag{11}$$

We remark here that the array (10) is said to be in row echelon form, while the array (11) is said to be in reduced row echelon form – precise definitions will follow in Sections 1.5–1.6. Let us see how we may solve the system (8) by using the arrays (10) or (11). First consider (10). Note that this represents the system

$$\begin{aligned} x_1 + 3x_2 + x_3 + 5x_4 + x_5 &= 5, \\ x_2 + x_3 + 2x_4 + x_5 &= 4, \\ x_4 + x_5 &= 1. \end{aligned} \tag{12}$$

First of all, take the third equation

$$x_4 + x_5 = 1.$$

If we let $x_5 = t$, then $x_4 = 1 - t$. Substituting these into the second equation, we obtain (you must do the calculation here)

$$x_2 + x_3 = 2 + t.$$

If we let $x_3 = s$, then $x_2 = 2 + t - s$. Substituting all these into the first equation, we obtain (you must do the calculation here)

$$x_1 = -6 + t + 2s.$$

Hence

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t)$$

is a solution of the system (12) for every $s, t \in \mathbb{R}$. In view of Proposition 1A, these are also precisely the solutions of the system (8). Alternatively, consider (11) instead. Note that this represents the system

$$\begin{aligned} x_1 \qquad - 2x_3 \qquad - x_5 &= -6, \\ x_2 + \ \ x_3 \qquad - x_5 &= \ \ 2, \\ x_4 + x_5 &= \ \ 1. \end{aligned} \tag{13}$$

First of all, take the third equation

$$x_4 + x_5 = 1.$$

If we let $x_5 = t$, then $x_4 = 1 - t$. Substituting these into the second equation, we obtain (you must do the calculation here)

$$x_2 + x_3 = 2 + t.$$

If we let $x_3 = s$, then $x_2 = 2 + t - s$. Substituting all these into the first equation, we obtain (you must do the calculation here)

$$x_1 = -6 + t + 2s.$$

Hence

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t)$$

is a solution of the system (13) for every $s, t \in \mathbb{R}$. In view of Proposition 1A, these are also precisely the solutions of the system (8). However, if you have done the calculations as suggested, you will notice that the calculation is easier for the system (13) than for the system (12). This is clearly a case of the array (11) in reduced row echelon form having more 0's than the array (10) in row echelon form, so that the system (13) has fewer non-zero coefficients than the system (12).

1.3. Row Echelon Form

Definition. A rectangular array of numbers is said to be in row echelon form if the following conditions are satisfied:

(1) The left-most non-zero entry of any non-zero row has value 1. These are called the pivot entries.
(2) All zero rows are grouped together at the bottom of the array.
(3) The pivot entry of a non-zero row occurring lower in the array is to the right of the pivot entry of a non-zero row occurring higher in the array.

Next, we investigate how we may reduce a given array to row echelon form. We shall illustrate the ideas by working on an example.

Example 1.3.1. Consider the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Step 1: Locate the left-most non-zero column and cover all columns to the left of this column (in our illustration here, $\times$ denotes an entry that has been covered). We now have

$$\begin{pmatrix} \times & 0 & 5 & 0 & 15 & 5 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Step 2: Consider the part of the array that remains uncovered. By interchanging rows if necessary, ensure that the top-left entry is non-zero. So let us interchange rows 1 and 4 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 3: If the top entry on the left-most uncovered column is a, then we multiply the top uncovered row by $1/a$ to ensure that this entry becomes 1. So let us divide row 1 by 1 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 2 & 4 & 7 & 1 & 3 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}!$$

Step 4: We now try to make all entries below the top entry on the left-most uncovered column zero. This can be achieved by adding suitable multiples of row 1 to the other rows. So let us add -2 times row 1 to row 2 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 1 & 2 & 3 & 0 & 1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Then let us add -1 times row 1 to row 3 to obtain

$$\begin{pmatrix} \times & 1 & 2 & 4 & 1 & 2 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 5: Now cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 0 & -1 & -1 & -1 \\ \times & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Step 6: Repeat Steps 1–5 on the uncovered array, and as many times as necessary so that eventually the whole array gets covered. So let us continue. Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Following Step 2, we interchanging rows if necessary to ensure that the top-left entry is non-zero. So let us interchange rows 1 and 3 (here we do not count any covered rows) to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 5 & 0 & 15 & 5 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 3, we multiply the top row by a suitable number to ensure that the top entry on the left-most uncovered column becomes 1. So let us multiply row 1 by $1/5$ to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & 1 & 0 & 3 & 1 \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 4, we do nothing! Following Step 5, we cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & 0 & -1 & -1 & -1 \\ \times & \times & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & -1 & -1 & -1 \\ \times & \times & \times & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 2, we do nothing! Following Step 3, we multiply the top row by a suitable number to ensure that the top entry on the left-most uncovered column becomes 1. So let us multiply row 1 by -1 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 1 & 1 & 1 \\ \times & \times & \times & -1 & -1 & -1 \end{pmatrix}.$$

Following Step 4, we now try to make all entries below the top entry on the left-most uncovered column zero. So let us add row 1 to row 2 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 1 & 1 & 1 \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}.$$

Following Step 5, we cover the top row. We then obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}.$$

Following Step 1, we locate the left-most non-zero column and cover all columns to the left of this column. We now have

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step ∞. Uncover everything! We then have

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in row echelon form.

In practice, we do not actually cover any entries of the array, so let us repeat here the same argument without covering anything – the reader is advised to compare this with the earlier discussion. We start with the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

Interchanging rows 1 and 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Adding -2 times row 1 to row 2, and adding -1 times row 1 to row 3, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 5 & 0 & 15 & 5 \end{pmatrix}.$$

Interchanging rows 2 and 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Multiplying row 1 by 1/5, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Multiplying row 3 by -1, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Adding row 3 to row 4, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in row echelon form.

Remarks. (1) As already observed earlier, we do not actually physically cover rows or columns. In any practical situation, we simply copy these entries without changes.

(2) The steps indicated the the first part of the last example are for guidance only. In practice, we do not have to follow the steps above religiously, and what we do is to a great extent dictated by good common sense. For instance, suppose that we are faced with the array

$$\begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 2 & 0 & 2 \end{pmatrix}.$$

If we follow the steps religiously, then we shall multiply row 1 by 1/2. However, note that this will introduce fractions to some entries of the array, and any subsequent calculation will become rather messy. Instead, let us multiply row 1 by 3 to obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 3 & 2 & 0 & 2 \end{pmatrix}.$$

Then let us multiply row 2 by 2 to obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 6 & 4 & 0 & 4 \end{pmatrix}.$$

Adding -1 times row 1 to row 2, we obtain

$$\begin{pmatrix} 6 & 9 & 6 & 3 \\ 0 & -5 & -6 & 1 \end{pmatrix}.$$

In this way, we have avoided the introduction of fractions until later in the process. In general, if we start with an array with integer entries, then it is possible to delay the introduction of fractions by omitting Step 3 until the very end.

Example 1.3.2. Consider the array

$$\begin{pmatrix} 2 & 1 & 3 & 2 & 5 \\ 1 & 3 & 2 & 4 & 1 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

Try following the steps indicated in the first part of the previous example religiously and try to see how complicated the calculations get. On the other hand, we can modify the steps with some common sense. First of all, we interchange rows 1 and 2 to obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 & 5 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

The reason for taking this step is to put an entry 1 at the top left without introducing fractions anywhere. When we next add multiples of row 1 to the other rows to make 0's below this 1, we do not introduce fractions either. Now adding -2 times row 1 to row 2, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & -5 & -1 & -6 & 3 \\ 3 & 2 & 0 & 0 & 2 \end{pmatrix}.$$

Adding -3 times row 1 to row 3, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & -5 & -1 & -6 & 3 \\ 0 & -7 & -6 & -12 & -1 \end{pmatrix}.$$

Next, multiplying row 2 by -7, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & -7 & -6 & -12 & -1 \end{pmatrix}.$$

Multiplying row 3 by -5, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 35 & 30 & 60 & 5 \end{pmatrix}.$$

Note that here we are essentially covering up row 1. Also, we have multiplied rows 2 and 3 by suitable multiples to that their leading non-zero entries are the same, in preparation for taking the next step without introducing fractions. Now adding -1 times row 2 to row 3, we obtain

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 0 & 23 & 18 & 26 \end{pmatrix}.$$

Here, the array is almost in row echelon form, except that the leading non-zero entries in rows 2 and 3 are not equal to 1. However, we can always multiply row 2 by $1/35$ and row 3 by $1/23$ if we want to obtain the row echelon form

$$\begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 1 & 1/5 & 6/5 & -3/5 \\ 0 & 0 & 1 & 18/23 & 26/23 \end{pmatrix}.$$

If this differs from the answer you got when you followed the steps indicated in the previous example religiously, do not worry. row echelon forms are not unique!

1.4. Reduced Row Echelon Form

Definition. A rectangular array of numbers is said to be in reduced row echelon form if the following conditions are satisfied:

(1) The left-most non-zero entry of any non-zero row has value 1. These are called the pivot entries.
(2) All zero rows are grouped together at the bottom of the array.
(3) The pivot entry of a non-zero row occurring lower in the array is to the right of the pivot entry of a non-zero row occurring higher in the array.
(4) Each column containing a pivot entry has 0's everywhere else in the column.

We now investigate how we may reduce a given array to reduced row echelon form. Here, we basically take an extra step to convert an array from row echelon form to reduced row echelon form. We shall illustrate the ideas by continuing on an earlier example.

Example 1.4.1. Consider again the array

$$\begin{pmatrix} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{pmatrix}.$$

We have already shown in Example 1.3.1 that this array can be reduced to row echelon form

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 1: Cover all zero rows at the bottom of the array. We now have

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 2: We now try to make all the entries above the pivot entry on the bottom row zero (here again we do not count any covered rows). This can be achieved by adding suitable multiples of the bottom row to the other rows. So let us add -4 times row 3 to row 1 to obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 3: Now cover the bottom row. We then obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step 4: Repeat Steps 2–3 on the uncovered array, and as many times as necessary so that eventually the whole array gets covered. So let us continue. Following Step 2, we add -2 times row 2 to row 1 to obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Following Step 3, we cover row 2 to obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Following Step 2, we do nothing! Following Step 3, we cover row 1 to obtain

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix}.$$

Step ∞. Uncover everything! We then have

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in reduced row echelon form.
Again, in practice, we do not actually cover any entries of the array, so let us repeat here the same argument without covering anything – the reader is advised to compare this with the earlier discussion. We start with the row echelon form

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding -4 times row 3 to row 1, we obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding -2 times row 2 to row 1, we obtain

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in reduced row echelon form.

1.5. Solving a System of Linear Equations

Let us first summarize what we have done so far. We study a system (1) of m linear equations in n variables $x_1, \ldots, x_n$. If we omit reference to the variables, then the system (1) can be represented by the array (2), with m rows and $n+1$ columns. We next reduce the array (2) to row echelon form or reduced row echelon form by elementary row operations.

By Proposition 1A, the system of linear equations represented by the array in row echelon form or reduced row echelon form has the same solution set as the system (1). It follows that to solve the system

(1), it remains to solve the system represented by the array in row echelon form or reduced row echelon form. We now describe a simple way to obtain all solutions of this system.

DEFINITION. Any column of an array (2) in row echelon form or reduced row echelon form containing a pivot entry is called a pivot column.

First of all, let us eliminate the situation when the system has no solutions. Suppose that the array (2) has been reduced to row echelon form, and that this contains a row of the form

$$\underbrace{0 \quad \ldots \quad 0}_{n} \qquad 1$$

corresponding to the last column of the array being a pivot column. This row represents the equation

$$0x_1 + \ldots + 0x_n = 1;$$

clearly the system cannot have any solution.

DEFINITION. Suppose that the array (2) in row echelon form or reduced row echelon form satisfies the condition that its last column is not a pivot column. Then any variable x_i corresponding to a pivot column is called a pivot variable. All other variables are called free variables.

EXAMPLE 1.5.1. Consider the array

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

representing the system

$$\begin{aligned} x_2 \qquad\qquad - 9x_5 &= -4, \\ x_3 \qquad + 3x_5 &= \ \ 1, \\ x_4 + \ \ x_5 &= \ \ 1. \end{aligned}$$

Note that the zero row in the array represents an equation which is trivial! Here the last column of the array is not a pivot column. Now columns $2, 3, 4$ are the pivot columns, so that x_2, x_3, x_4 are the pivot variables and x_1, x_5 are the free variables.

To solve the system, we allow the free variables to take any values we choose, and then solve for the pivot variables in terms of the values of these free variables.

EXAMPLE 1.5.2. Consider the system of 4 linear equations

$$\begin{aligned} 5x_3 \qquad\quad + 15x_5 &= 5, \\ 2x_2 + 4x_3 + 7x_4 + \quad x_5 &= 3, \\ x_2 + 2x_3 + 3x_4 \qquad\quad &= 1, \\ x_2 + 2x_3 + 4x_4 + \quad x_5 &= 2, \end{aligned} \tag{14}$$

in the 5 variables x_1, x_2, x_3, x_4, x_5. If we omit reference to the variables, then the system can be represented by the array

$$\left(\begin{array}{ccccc|c} 0 & 0 & 5 & 0 & 15 & 5 \\ 0 & 2 & 4 & 7 & 1 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 & 1 & 2 \end{array}\right). \tag{15}$$

As in Example 1.3.1, we can reduce the array (15) to row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & 4 & 1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{16}$$

representing the system

$$\begin{aligned} x_2 + 2x_3 + 4x_4 + x_5 &= 2, \\ x_3 + 3x_5 &= 1, \\ x_4 + x_5 &= 1. \end{aligned} \tag{17}$$

Alternatively, as in Example 1.4.1, we can reduce the array (15) to reduced row echelon form

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -9 & -4 \\ 0 & 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{18}$$

representing the system

$$\begin{aligned} x_2 - 9x_5 &= -4, \\ x_3 + 3x_5 &= 1, \\ x_4 + x_5 &= 1. \end{aligned} \tag{19}$$

By Proposition 1A, the three systems (14), (17) and (19) have exactly the same solution set. Now, we observe from (16) or (18) that columns $2, 3, 4$ are the pivot columns, so that x_2, x_3, x_4 are the pivot variables and x_1, x_5 are the free variables. If we assign values $x_1 = s$ and $x_5 = t$, then we have, from (17) (harder) or (19) (easier), that

$$(x_1, x_2, x_3, x_4, x_5) = (s, 9t - 4, -3t + 1, -t + 1, t). \tag{20}$$

It follows that (20) is a solution of the system (14) for every $s, t \in \mathbb{R}$.

Example 1.5.3. Let us return to Example 1.2.4, and consider again the system (8) of 3 linear equations in the 5 variables x_1, x_2, x_3, x_4, x_5. If we omit reference to the variables, then the system can be represented by the array (9). We can reduce the array (9) to row echelon form (10), representing the system (12). Alternatively, we can reduce the array (9) to reduced row echelon form (11), representing the system (13). By Proposition 1A, the three systems (8), (12) and (13) have exactly the same solution set. Now, we observe from (10) or (11) that columns $1, 2, 4$ are the pivot columns, so that x_1, x_2, x_4 are the pivot variables and x_3, x_5 are the free variables. If we assign values $x_3 = s$ and $x_5 = t$, then we have, from (12) (harder) or (13) (easier), that

$$(x_1, x_2, x_3, x_4, x_5) = (-6 + t + 2s, 2 + t - s, s, 1 - t, t). \tag{21}$$

It follows that (21) is a solution of the system (8) for every $s, t \in \mathbb{R}$.

Example 1.5.4. In this example, we do not bother even to reduce the matrix to row echelon form. Consider the system of 3 linear equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + 2x_4 &= 5, \\ x_1 + 3x_2 + 2x_3 + 4x_4 &= 1, \\ 3x_1 + 2x_2 &= 2, \end{aligned} \tag{22}$$

in the 4 variables x_1, x_2, x_3, x_4. If we omit reference to the variables, then the system can be represented by the array

$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 5 \\ 1 & 3 & 2 & 4 & 1 \\ 3 & 2 & 0 & 0 & 2 \end{array}\right). \tag{23}$$

As in Example 1.3.2, we can reduce the array (23) to the form

$$\left(\begin{array}{cccc|c} 1 & 3 & 2 & 4 & 1 \\ 0 & 35 & 7 & 42 & -21 \\ 0 & 0 & 23 & 18 & 26 \end{array}\right), \tag{24}$$

representing the system

$$\begin{aligned} x_1 + \ 3x_2 + \ 2x_3 + \ 4x_4 &= \quad 1, \\ 35x_2 + \ 7x_3 + 42x_4 &= -21, \\ 23x_3 + 18x_4 &= \quad 26. \end{aligned} \tag{25}$$

Note that the array (24) is almost in row echelon form, except that the pivot entries are not 1. By Proposition 1A, the two systems (22) and (25) have exactly the same solution set. Now, we observe from (24) that columns $1, 2, 3$ are the pivot columns, so that x_1, x_2, x_3 are the pivot variables and x_4 is the free variable. If we assign values $x_4 = s$, then we have, from (25), that

$$(x_1, x_2, x_3, x_4) = \left(\frac{16}{23}s + \frac{28}{23}, -\frac{24}{23}s - \frac{19}{23}, -\frac{18}{23}s + \frac{26}{23}, s\right). \tag{26}$$

It follows that (26) is a solution of the system (22) for every $s \in \mathbb{R}$.

1.6. Homogeneous Systems

Consider a homogeneous system of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= 0, \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= 0, \\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0, \end{aligned} \tag{27}$$

with n variables $x_1, x_2, \ldots, x_n$. If we omit reference to the variables, then system (27) can be represented by the array

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 \end{array}\right) \tag{28}$$

of all the coefficients.

Note that the system (27) always has a solution, namely the trivial solution

$$x_1 = x_2 = \ldots = x_n = 0.$$

Indeed, if we reduce the array (28) to row echelon form or reduced row echelon form, then it is not difficult to see that the last column is a zero column and so cannot be a pivot column.

On the other hand, if the system (27) has a non-trivial solution, then we can multiply this solution by any non-zero real number different from 1 to obtain another non-trivial solution. We have therefore proved the following simple result.

PROPOSITION 1B. *The homogeneous system* (27) *either has the trivial solution as its only solution or has infinitely many solutions.*

The purpose of this section is to discuss the following stronger result.

PROPOSITION 1C. *Suppose that the system* (27) *has more variables than equations; in other words, suppose that* $n > m$. *Then there are infinitely many solutions.*

To see this, let us consider the array (28) representing the system (27). Note that (28) has m rows, corresponding to the number of equations. Also (28) has $n + 1$ columns, where n is the number of variables. However, the column of (28) on the extreme right is a zero column, corresponding to the fact that the system is homogeneous. Furthermore, this column remains a zero column if we perform elementary row operations on the array (28). If we now reduce (28) to row echelon form by elementary row operations, then there are at most m pivot columns, since there are only m equations in (27) and m rows in (28). It follows that if we exclude the zero column on the extreme right, then the remaining n columns cannot all be pivot columns. Hence at least one of the variables is a free variable. By assigning this free variable arbitrary real values, we end up with infinitely many solutions for the system (27).

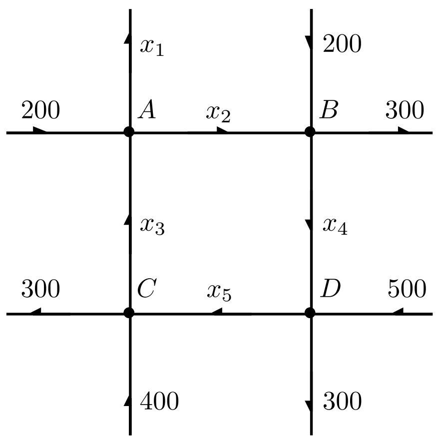
1.7. Application to Network Flow

Systems of linear equations arise when we investigate the flow of some quantity through a network. Such networks arise in science, engineering and economics. Two such examples are the pattern of traffic flow through a city and distribution of products from manufacturers to consumers through a network of wholesalers and retailers.

A network consists of a set of points, called the nodes, and directed lines connecting some or all of the nodes. The flow is indicated by a number or a variable. We observe the following basic assumptions:

- The total flow into a node is equal to the total flow out of a node.
- The total flow into the network is equal to the total flow out of the network.

Example 1.7.1. The picture below represents a system of one way streets in a particular part of some city and the traffic flow along the streets between the junctions:



We first equate the total flow into each node with the total flow out of the same node:

$$\begin{array}{ll} \text{node } A: & 200 + x_3 = x_1 + x_2, \\ \text{node } B: & 200 + x_2 = 300 + x_4, \\ \text{node } C: & 400 + x_5 = 300 + x_3, \\ \text{node } D: & 500 + x_4 = 300 + x_5. \end{array}$$

We then equate the total flow into and out of the network:

$$400 + 200 + 200 + 500 = 300 + 300 + x_1 + 300.$$

These give rise to a system of 5 linear equations

$$\begin{array}{rcl} x_1 + x_2 - x_3 \qquad\qquad & = & 200, \\ x_2 \qquad - x_4 \qquad & = & 100, \\ x_3 \qquad - x_5 & = & 100, \\ x_4 - x_5 & = & -200, \\ x_1 \qquad\qquad\qquad\qquad & = & 400, \end{array}$$

in the 5 variables $x_1, \ldots, x_5$, with augmented matrix

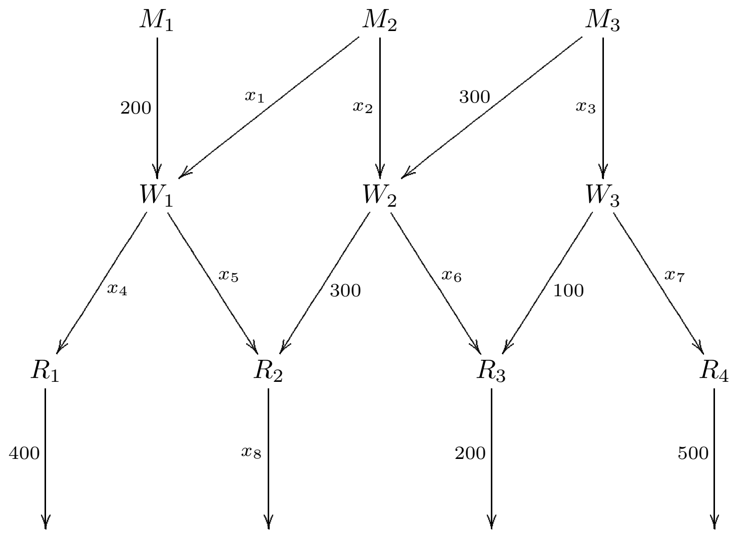
$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 1 & 0 & 0 & 0 & 0 & 400 \end{array} \right).$$

This has reduced row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 1 & 0 & -1 & 0 & 100 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & -200 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

We have general solution $(x_1, \ldots, x_5) = (400, t - 100, t + 100, t - 200, t)$, where t is a parameter. Since one way streets do not permit negative flow, all the coordinates have to be non-negative. It follows that $t \geq 200$.

Example 1.7.2. The picture below represents the quantities of a particular product that flow from manufacturers M_1, M_2, M_3, through wholesalers W_1, W_2, W_3 and retailers R_1, R_2, R_3, R_4, to consumers:



We first equate the total flow into each node with the total flow out of the same node:

$$\begin{array}{ll} \text{node } W_1: & 200 + x_1 = x_4 + x_5, \\ \text{node } W_2: & 300 + x_2 = 300 + x_6, \\ \text{node } W_3: & x_3 = 100 + x_7, \\ \text{node } R_1: & x_4 = 400, \\ \text{node } R_2: & 300 + x_5 = x_8, \\ \text{node } R_3: & 100 + x_6 = 200, \\ \text{node } R_4: & x_7 = 500. \end{array}$$

We then equate the total flow into and out of the network:

$$200 + x_1 + x_2 + 300 + x_3 = 400 + x_8 + 200 + 500.$$

These give rise to a system of 8 linear equations

$$\begin{array}{rcl} x_1 - x_4 - x_5 & = & -200, \\ x_2 - x_6 & = & 0, \\ x_3 - x_7 & = & 100, \\ x_4 & = & 400, \\ x_5 - x_8 & = & -300, \\ x_6 & = & 100, \\ x_7 & = & 500, \\ x_1 + x_2 + x_3 - x_8 & = & 600, \end{array}$$

in the 8 variables $x_1, \ldots, x_8$, with augmented matrix

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -300 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 500 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 600 \end{array}\right).$$

This has row echelon form

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 400 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -300 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We have general solution $(x_1, \ldots, x_8) = (t-100, 100, 600, 400, t-300, 100, 500, t)$, where t is a parameter. If no goods is returned, then all the coordinates have to be non-negative. It follows that $t \geq 300$.

1.8. Application to Electrical Networks

A simple electric circuit consists of two basic components, electrical sources where the electrical potential E is measured in volts (V), and resistors where the resistence R is measured in ohms (Ω). We are interested in determining the current I measured in amperes (A).

The electrical potential between two points is sometimes called the voltage drop between these two points. Currents and voltage drops can be positive or negative.

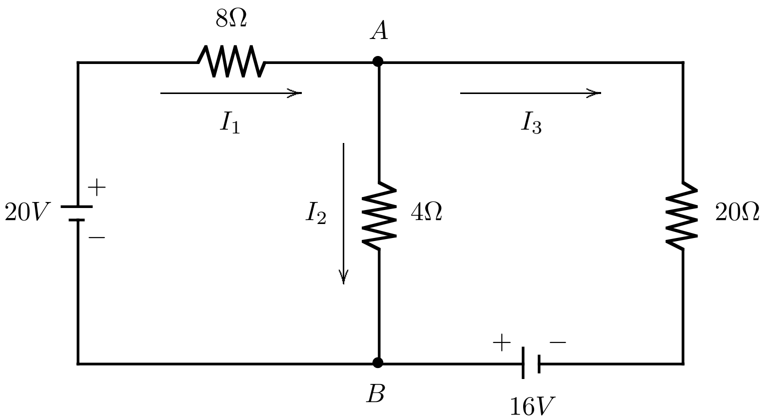
The current flow in an electrical circuit is governed by three basic rules:

- Ohm's law: The voltage drop E across a resistor with resistence R with a current I passing through it is given by $E = IR$.
- Current law: The sum of the currents flowing into any point is the same as the sum of the currents flowing out of the point.
- Voltage law: The sum of the voltage drops around any closed loop is equal to zero.

Around any loop, we select a positive direction – clockwise or anticlockwise as we see fit. We have the following convention:

- The voltage drop across a resistor is taken to be positive if the current flows in the positive direction of the loop, and negative if the current flows in the negative direction of the loop.
- The voltage drop across an electrical source is taken to be positive if the positive direction of the loop is from + to −, and negative if the positive direction of the loop is from − to +.

EXAMPLE 1.8.1. Consider the electric circuit shown in the diagram below:



We wish to determine the currents I_1, I_2 and I_3. Applying the Current law to the point A, we obtain $I_1 = I_2 + I_3$. Applying the Current law to the point B, we obtain the same. Hence we have the linear equation

$$I_1 - I_2 - I_3 = 0.$$

Next, let us consider the left hand loop, and let us take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 8Ω resistor is $8I_1$, while the voltage drop across the 4Ω resistor is $4I_2$. On the other hand, the voltage drop across the $20V$ electrical source is negative, since the positive direction of the loop is from − to +. The Voltage law applied to this loop now gives $8I_1 + 4I_2 - 20 = 0$, and we have the linear equation

$$8I_1 + 4I_2 = 20, \qquad \text{or} \qquad 2I_1 + I_2 = 5.$$

Next, let us consider the right hand loop, and let us take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 20Ω resistor is $20I_3$, while the voltage drop across the 4Ω resistor is $-4I_2$. On the other hand, the voltage drop across the $16V$ electrical source is negative, since the positive direction of the loop is from − to +. The Voltage law applied to this loop now gives $20I_3 - 4I_2 - 16 = 0$, and we have the linear equation

$$4I_2 - 20I_3 = -16, \qquad \text{or} \qquad I_2 - 5I_3 = -4.$$

We now have a system of three linear equations

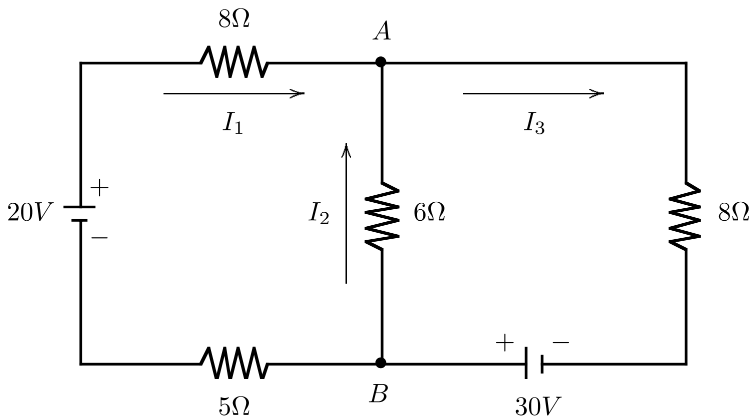
$$\begin{aligned} I_1 - I_2 - \ \ I_3 &= \ \ 0, \\ 2I_1 + I_2 \qquad &= \ \ 5, \\ I_2 - 5I_3 &= -4. \end{aligned} \tag{29}$$

The augmented matrix is given by

$$\begin{pmatrix} 1 & -1 & -1 & \vert & 0 \\ 2 & 1 & 0 & \vert & 5 \\ 0 & 1 & -5 & \vert & -4 \end{pmatrix}, \qquad \text{with reduced row echelon form} \qquad \begin{pmatrix} 1 & 0 & 0 & \vert & 2 \\ 0 & 1 & 0 & \vert & 1 \\ 0 & 0 & 1 & \vert & 1 \end{pmatrix}.$$

Hence $I_1 = 2$ and $I_2 = I_3 = 1$. Note here that we have not considered the outer loop. Suppose again that we take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 8Ω resistor is $8I_1$, while the voltage drop across the 20Ω resistor is $20I_3$. On the other hand, the voltage drop across the $20V$ and $16V$ electrical sources are both negative. The Voltage law applied to this loop then gives $8I_1 + 20I_3 - 36 = 0$. But this equation can be obtained by combining the last two equations in (29).

Example 1.8.2. Consider the electric circuit shown in the diagram below:



We wish to determine the currents I_1, I_2 and I_3. Applying the Current law to the point A, we obtain $I_1 + I_2 = I_3$. Applying the Current law to the point B, we obtain the same. Hence we have the linear equation

$$I_1 + I_2 - I_3 = 0.$$

Next, let us consider the left hand loop, and let us take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 8Ω resistor is $8I_1$, the voltage drop across the 6Ω resistor is $-6I_2$, while the voltage drop across the 5Ω resistor is $5I_1$. On the other hand, the voltage drop across the $20V$ electrical source is negative, since the positive direction of the loop is from $-$ to $+$. The Voltage law applied to this loop now gives $8I_1 - 6I_2 + 5I_1 - 20 = 0$, and we have the linear equation

$$13I_1 - 6I_2 = 20.$$

Next, let us consider the outer loop, and let us take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 8Ω resistor on the top is $8I_1$, the voltage drop across the 8Ω resistor on the right is $8I_3$, while the voltage drop across the 5Ω resistor is $5I_1$. On the other hand, the voltage drop across the $30V$ and $20V$ electrical sources are both negative, since the positive direction of the loop is from $-$ to $+$ in each case. The Voltage law applied to this loop now gives $8I_1 + 8I_3 + 5I_1 - 50 = 0$, and we have the linear equation

$$13I_1 + 8I_3 = 50.$$

We now have a system of three linear equations

$$\begin{aligned} I_1 + I_2 - I_3 &= 0, \\ 13I_1 - 6I_2 \qquad &= 20, \\ 13I_1 \qquad + 8I_3 &= 50. \end{aligned} \tag{30}$$

The augmented matrix is given by

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 13 & -6 & 0 & 20 \\ 13 & 0 & 8 & 50 \end{array}\right), \qquad \text{with reduced row echelon form} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Hence $I_1 = 2$, $I_2 = 1$ and $I_3 = 3$. Note here that we have not considered the right hand loop. Suppose again that we take the positive direction to be clockwise. By Ohm's law, the voltage drop across the 8Ω resistor is $8I_3$, while the voltage drop across the 6Ω resistor is $6I_2$. On the other hand, the voltage drop across the $30V$ electrical source is negative. The Voltage law applied to this loop then gives $8I_3 + 6I_2 - 30 = 0$. But this equation can be obtained by combining the last two equations in (30).

1.9. Application to Economics

In this section, we describe a simple exchange model due to the economist Leontief. An economy is divided into sectors. We know the total output for each sector as well as how outputs are exchanged among the sectors. The value of the total output of a given sector is known as the price of the output.

Leontief has shown that there exist equilibrium prices that can be assigned to the total output of the sectors in such a way that the income for each sector is exactly the same as its expenses.

EXAMPLE 1.9.1. An economy consists of three sectors A, B, C which purchase from each other according to the table below:

	proportion of output from sector A	B	C
purchased by sector A	0.2	0.6	0.1
purchased by sector B	0.4	0.1	0.5
purchased by sector C	0.4	0.3	0.4

Let p_A, p_B, p_C denote respectively the value of the total output of sectors A, B, C. For the expense to match the value for each sector, we must have

$$\begin{aligned} 0.2p_A + 0.6p_B + 0.1p_C &= p_A, \\ 0.4p_A + 0.1p_B + 0.5p_C &= p_B, \\ 0.4p_A + 0.3p_B + 0.4p_C &= p_C, \end{aligned}$$

leading to the homogeneous linear equations

$$\begin{aligned} 0.8p_A - 0.6p_B - 0.1p_C &= 0, \\ 0.4p_A - 0.9p_B + 0.5p_C &= 0, \\ 0.4p_A + 0.3p_B - 0.6p_C &= 0, \end{aligned}$$

giving rise to the augmented matrix

$$\left(\begin{array}{ccc|c} 0.8 & -0.6 & -0.1 & 0 \\ 0.4 & -0.9 & 0.5 & 0 \\ 0.4 & 0.3 & -0.6 & 0 \end{array}\right), \qquad \text{or simply} \qquad \left(\begin{array}{ccc|c} 8 & -6 & -1 & 0 \\ 4 & -9 & 5 & 0 \\ 4 & 3 & -6 & 0 \end{array}\right).$$

This can be reduced by elementary row operations to

$$\left(\begin{array}{ccc|c} 16 & 0 & -13 & 0 \\ 0 & 12 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

leading to the solution $(p_A, p_B, p_C) = t(\frac{13}{16}, \frac{11}{12}, 1)$ if we assign the free variable $p_C = t$, or to the solution $(p_A, p_B, p_C) = t(39, 44, 48)$ if we assign the free variable $p_C = 48t$, where t is a real parameter. For the latter, the choice $t = 10^6$ gives rise to the prices of 39, 44 and 48 million for the three sectors A, B, C respectively.

1.10. Application to Chemistry

Chemical equations consist of reactants and products. The problem is to balance such equations so that the following two rules apply:

- Conservation of mass: No atoms are produced or destroyed in a chemical reaction.
- Conservation of charge: The total charge of reactants is equal to the total charge of the products.

EXAMPLE 1.10.1. Consider the oxidation of ammonia to form nitric oxide and water, given by the chemical equation

$$(x_1)NH_3 + (x_2)O_2 \longrightarrow (x_3)NO + (x_4)H_2O.$$

Here the reactants are ammonia (NH_3) and oxygen (O_2), while the products are nitric oxide (NO) and water (H_2O). Our problem is to find the smallest positive integer values of x_1, x_2, x_3, x_4 such that the equation balances. To do this, the technique is to equate the total number of each type of atoms on the two sides of the chemical equation:

$$\begin{array}{ll} \text{atom } N: & x_1 = x_3, \\ \text{atom } H: & 3x_1 = 2x_4, \\ \text{atom } O: & 2x_2 = x_3 + x_4. \end{array}$$

These give rise to a homogeneous system of 3 linear equations

$$\begin{aligned} x_1 \qquad - x_3 \qquad &= 0, \\ 3x_1 \qquad\qquad - 2x_4 &= 0, \\ 2x_2 - x_3 - \ x_4 &= 0, \end{aligned}$$

in the 4 variables $x_1, \ldots, x_4$, with augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right),$$

which can be simplified by elementary row operations to

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right),$$

leading to the general solution $(x_1, \ldots, x_4) = t(\frac{2}{3}, \frac{5}{6}, \frac{2}{3}, 1)$ if we assign the free variable $x_4 = t$. The choice $t = 6$ gives rise to the smallest positive integer solution $(x_1, \ldots, x_4) = (4, 5, 4, 6)$, leading to the balanced chemical equation

$$4NH_3 + 5O_2 \longrightarrow 4NO + 6H_2O.$$

EXAMPLE 1.10.2. Consider the chemical equation

$$(x_1)CO + (x_2)CO_2 + (x_3)H_2 \longrightarrow (x_4)CH_4 + (x_5)H_2O.$$

We equate the total number of each type of atoms on the two sides of the chemical equation:

$$\begin{array}{ll} \text{atom } C: & x_1 + x_2 = x_4, \\ \text{atom } O: & x_1 + 2x_2 = x_5, \\ \text{atom } H: & 2x_3 = 4x_4 + 2x_5. \end{array}$$

These give rise to a homogeneous system of 3 linear equations

$$\begin{aligned} x_1 + x_2 - x_4 &= 0, \\ x_1 + 2x_2 - x_5 &= 0, \\ 2x_3 - 4x_4 - 2x_5 &= 0, \end{aligned}$$

in the 5 variables $x_1, \ldots, x_5$, with augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -4 & -2 & 0 \end{array}\right),$$

with reduced row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \end{array}\right),$$

leading to the general solution $(x_1, \ldots, x_5) = s(2, -1, 2, 1, 0) + t(-1, 1, 1, 0, 1)$ if we assign the two free variables $x_4 = s$ and $x_5 = t$. The choice $s = 2$ and $t = 3$ leads to the solution $(x_1, \ldots, x_5) = (1, 1, 7, 2, 3)$, with balanced chemical equation

$$CO + CO_2 + 7H_2 \longrightarrow 2CH_4 + 3H_2O;$$

the choice $s = 3$ and $t = 4$ leads to the solution $(x_1, \ldots, x_5) = (2, 1, 10, 3, 4)$, with balanced chemical equation

$$2CO + CO_2 + 10H_2 \longrightarrow 3CH_4 + 4H_2O;$$

while the choice $s = 3$ and $t = 5$ leads to the solution $(x_1, \ldots, x_5) = (1, 2, 11, 3, 5)$, with balanced chemical equation

$$CO + 2CO_2 + 11H_2 \longrightarrow 3CH_4 + 5H_2O.$$

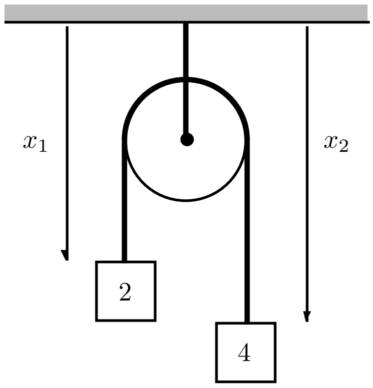
All these are known to happen.

1.11. Application to Mechanics

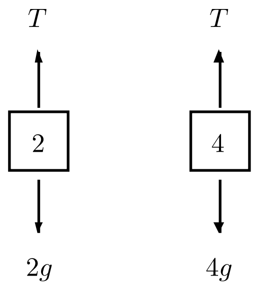
In this section, we consider the problem of systems of weights, light ropes and smooth light pulleys, subject to the following two main principles:

- If a light rope passes around one or more smooth light pulleys, then the tension at the two ends are the same.
- Newton's second law of motion: We have $F = m\ddot{x}$, where F denotes force, m denotes mass and $\ddot{x}$ denotes acceleration.

EXAMPLE 1.11.1. Two particles, of mass 2 and 4 (kilograms), are attached to the ends of a light rope passing around a smooth light pulley suspended from the ceiling as shown in the diagram below:



We would like to find the tension in the rope and the acceleration of each particle. Here it will be convenient that the distances x_1 and x_2 are measured downwards, and we take this as the positive direction, so that any positive accelaration is downwards. We first apply Newton's law of motion to each particle. The picture below summarizes the forces action on the two particles:



Here T denotes the tension in the rope, and g denotes acceleration due to gravity. Newton's law of motion applied to the two particles (downwards) then give the equations

$$2\ddot{x}_1 = 2g - T \qquad \text{and} \qquad 4\ddot{x}_2 = 4g - T.$$

We also have the conservation of the length of the rope, in the form $x_1 + x_2 = C$, so that $\ddot{x}_1 + \ddot{x}_2 = 0$. To summarize, for the three variables $\ddot{x}_1, \ddot{x}_2, T$, we have the system of linear equations

$$\begin{aligned} 2\ddot{x}_1 \qquad\quad + T &= 2g, \\ 4\ddot{x}_2 + T &= 4g, \\ \ddot{x}_1 + \ddot{x}_2 \qquad &= 0, \end{aligned}$$

with augmented matrix

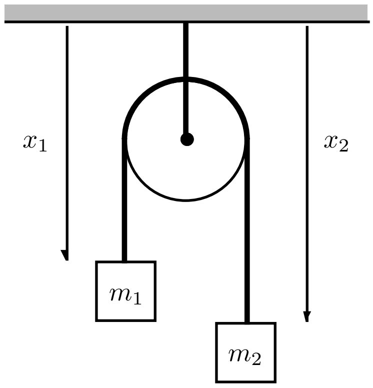
$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2g \\ 0 & 4 & 1 & 4g \\ 1 & 1 & 0 & 0 \end{array}\right),$$

which can be reduced by elementary row operations to

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 2g \\ 0 & 0 & 3 & 8g \end{array}\right).$$

This leads to the solution $(\ddot{x}_1, \ddot{x}_2, T) = (-\frac{1}{3}g, \frac{1}{3}g, \frac{8}{3}g)$.

EXAMPLE 1.11.2. We now generalize the problem in the previous example. Two particles, of mass m_1 and m_2, are attached to the ends of a light rope passing around a smooth light pulley suspended from the ceiling as shown in the diagram below:



For the three variables $\ddot{x}_1, \ddot{x}_2, T$, we now have the system of linear equations

$$\begin{aligned} m_1\ddot{x}_1 \qquad\quad + T &= m_1 g, \\ m_2\ddot{x}_2 + T &= m_2 g, \\ \ddot{x}_1 + \quad \ddot{x}_2 \qquad &= \quad 0, \end{aligned}$$

with augmented matrix

$$\left(\begin{array}{ccc|c} m_1 & 0 & 1 & m_1 g \\ 0 & m_2 & 1 & m_2 g \\ 1 & 1 & 0 & 0 \end{array} \right),$$

which can be reduced by elementary row operations to

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & m_1 m_2 & m_1 & m_1 m_2 g \\ 0 & 0 & m_1 + m_2 & 2m_1 m_2 g \end{array} \right).$$

This leads to the solution

$$(\ddot{x}_1, \ddot{x}_2, T) = \left(\frac{m_1 - m_2}{m_1 + m_2} g, \frac{m_2 - m_1}{m_1 + m_2} g, \frac{2m_1 m_2}{m_1 + m_2} g \right).$$

Note that if $m_1 = m_2$, then $\ddot{x}_1 = \ddot{x}_2 = 0$, so that the particles are stationary. On the other hand, if $m_2 > m_1$, then $\ddot{x}_2 > 0$ and $\ddot{x}_1 < 0$. Then

$$T < \frac{2m_1 m_2}{m_1 + m_1} g = m_2 g \qquad \text{and} \qquad T > \frac{2m_1 m_2}{m_2 + m_2} g = m_1 g.$$

Hence $m_1 g < T < m_2 g$.

Problems for Chapter 1

1. Consider the system of linear equations

$$\begin{aligned} 2x_1 + 5x_2 + 8x_3 &= 2, \\ x_1 + 2x_2 + 3x_3 &= 4, \\ 3x_1 + 4x_2 + 4x_3 &= 1. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

2. Consider the system of linear equations

$$\begin{aligned} 4x_1 + 5x_2 + 8x_3 &= 0, \\ x_1 \qquad\quad + 3x_3 &= 6, \\ 3x_1 + 4x_2 + 6x_3 &= 9. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

3. Consider the system of linear equations

$$\begin{aligned} x_1 - x_2 - 7x_3 + 7x_4 &= 5, \\ -x_1 + x_2 + 8x_3 - 5x_4 &= -7, \\ 3x_1 - 2x_2 - 17x_3 + 13x_4 &= 14, \\ 2x_1 - x_2 - 11x_3 + 8x_4 &= 7. \end{aligned}$$

   a) Write down the augmented matrix for this system.
   b) Reduce the augmented matrix by elementary row operations to row echelon form.
   c) Use your answer in part (b) to solve the system of linear equations.

4. Solve the system of linear equations

$$\begin{aligned} x + 3y - 2z &= 4, \\ 2x + 7y + 2z &= 10. \end{aligned}$$

5. For each of the augmented matrices below, reduce the matrix to row echelon or reduced row echelon form, and solve the system of linear equations represented by the matrix:

   a) $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 3 & 2 & -1 & 3 & 6 \\ 4 & 3 & 1 & 4 & 11 \\ 2 & 1 & -3 & 2 & 1 \end{array}\right)$
   b) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 1 \\ 2 & -5 & -3 & 12 & 2 \\ 7 & 1 & 8 & 5 & 7 \end{array}\right)$

6. Reduce each of the following arrays by elementary row operations to reduced row echelon form:

   a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}$
   b) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
   c) $\begin{pmatrix} 1 & 11 & 21 & 31 & 41 & 51 \\ 2 & 12 & 22 & 32 & 42 & 52 \\ 3 & 13 & 23 & 33 & 43 & 53 \end{pmatrix}$

7. Consider a system of linear equations in five variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and expressed in matrix form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is written as a column matrix. Suppose that the augmented matrix $(A|\mathbf{b})$ can be reduced by elementary row operations to the row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 3 & 2 & 0 & 6 & 4 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

   a) Which are the pivot variables and which are the free variables?
   b) Determine all the solutions of the system of linear equations.

8. Consider a system of linear equations in five variables $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ and expressed in matrix form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is written as a column matrix. Suppose that the augmented matrix $(A|\mathbf{b})$ can be reduced by elementary row operations to the row echelon form

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

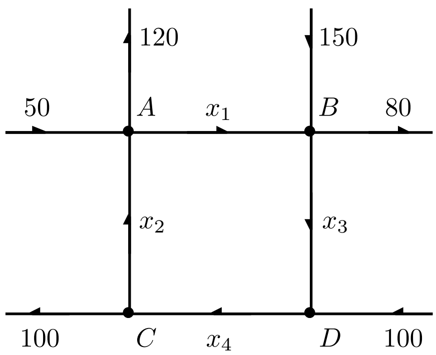
   a) Which are the pivot variables and which are the free variables?
   b) Determine all the solutions of the system of linear equations.

9. Consider the system of linear equations

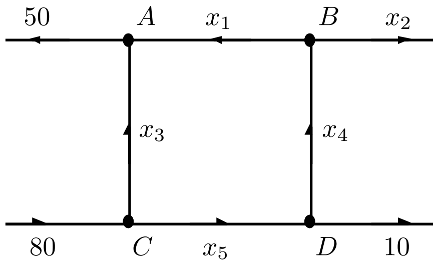
$$\begin{aligned} x_1 + \lambda x_2 - x_3 &= 1, \\ 2x_1 + x_2 + 2x_3 &= 5\lambda + 1, \\ x_1 - x_2 + 3x_3 &= 4\lambda + 2, \\ x_1 - 2\lambda x_2 + 7x_3 &= 10\lambda - 1. \end{aligned}$$

   a) Reduce its associated augmented matrix to row echelon form.
      [HINT: After one or two steps, we will find the calculations extremely unpleasant, particularly since we do not know whether λ is zero or non-zero. Try rewriting the system of equations as a system in the variables x_1, x_3, x_2, so that columns 2 and 3 of the augmented matrix are now swapped.]
   b) Find a value of λ for which the system is soluble.
   c) Solve the system.

10. Find the minimum value for x_4 in the following system of one way streets:

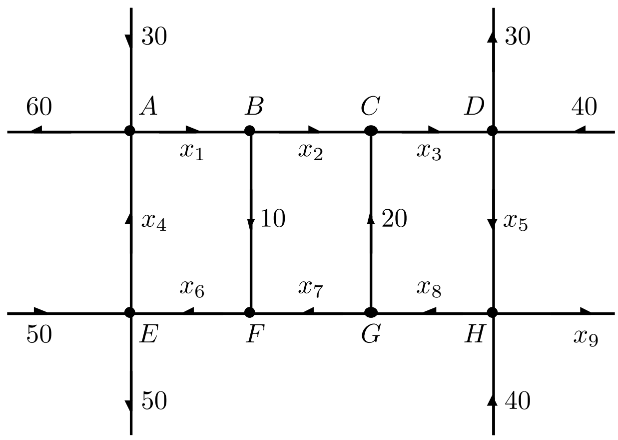


11. Consider the traffic flow in the following system of one way streets:



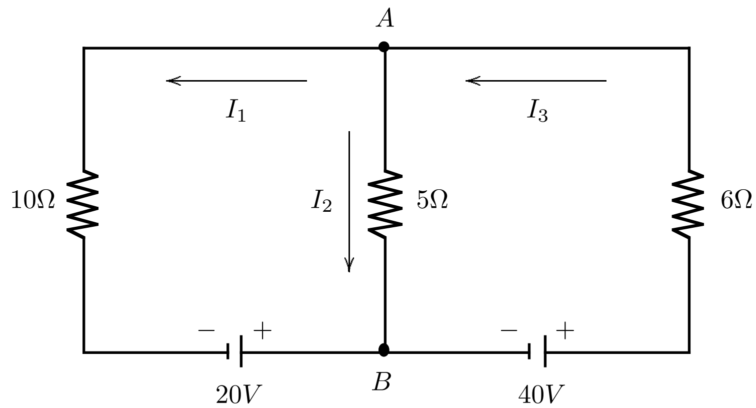
a) Find the general solution of the system.
b) Find the range for x_5, and then determine the range for each of the other four variables.

12. Consider the traffic flow in the following system of one way streets:



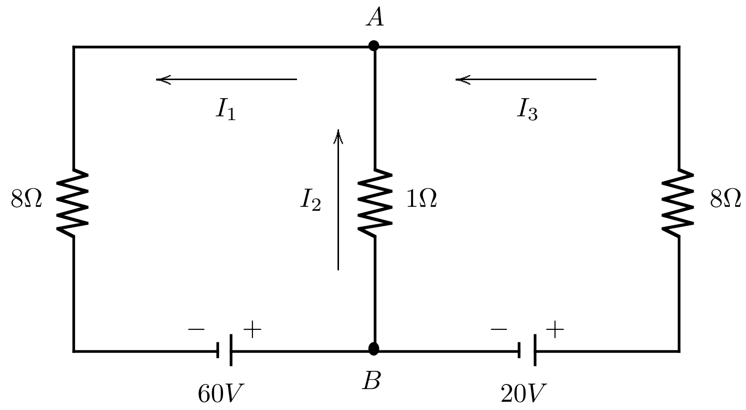
a) Find the general solution of the system.
b) Find the range for x_8, and then determine the range for each of the other eight variables.

13. Consider the electric circuit shown in the diagram below:



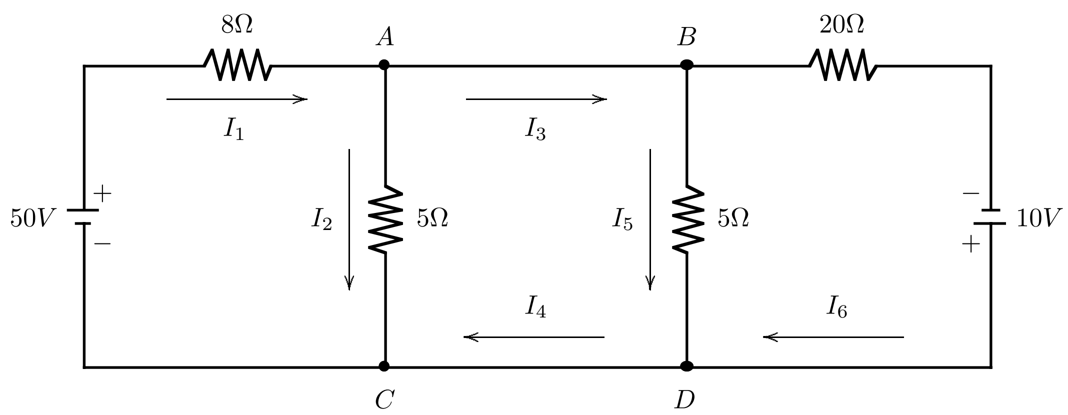
Determine the currents I_1, I_2 and I_3. You must explain each step carefully, quoting all the relevant laws on electric circuits. In particular, you must clearly indicate the positive direction of each loop you are considering, and ensure that the voltage drop across every resistor and electrical source on the loop carries the correct sign.

14. Consider the electric circuit shown in the diagram below:



Determine the currents I_1, I_2 and I_3. You must explain each step carefully, quoting all the relevant laws on electric circuits. In particular, you must clearly indicate the positive direction of each loop you are considering, and ensure that the voltage drop across every resistor and electrical source on the loop carries the correct sign.

15. Consider the electric circuit shown in the diagram below:



Determine the currents I_1, I_2, I_3, I_4, I_5 and I_6. You must explain each step carefully, quoting all the relevant laws on electric circuits. In particular, you must clearly indicate the positive direction of each loop you are considering, and ensure that the voltage drop across every resistor and electrical source on the loop carries the correct sign.

16. Three industries A, B, C consume their own outputs and also buy from each other according to the table below:

	proportion of output of industry A	B	C
bought by industry A	0.35	0.50	0.30
bought by industry B	0.25	0.20	0.30
bought by industry C	0.40	0.30	0.40

Use the simple exchange model due to the economist Leontief to determine equilibrium prices that they can charge each other so that no money changes hands.

17. An arrangement exists for three colleagues A, B, C who work for themselves and each other according to the table below:

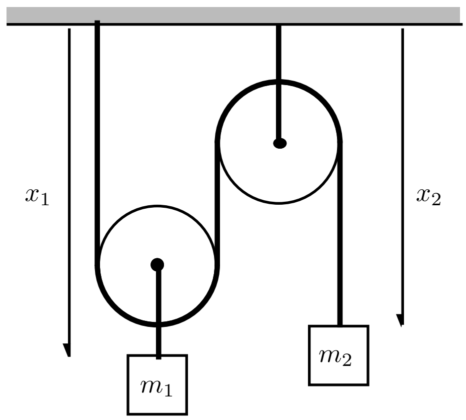
	percentage of time spent by A	B	C
working for A	50	40	10
working for B	10	20	60
working for C	40	40	30

Use the simple exchange model due to the economist Leontief to determine equilibrium fees that they can charge each other so that no money changes hands.

18. Three farmers A, B, C grow bananas, oranges and apples respectively, and buy off each other. Farmer A buys 50% of the oranges and 20% of the apples, farmer B buys 30% of the bananas and 40% of the apples, while farmer C buys 50% of the bananas and 20% of the oranges. Use the simple exchange model due to the economist Leontief to determine equilibrium prices that they can charge each other so that no money changes hands.

19. For each of the following chemical reactions, determine the balanced chemical equation:
a) reactants Al and O_2; product Al_2O_3
b) reactants C_2H_6 and O_2; products CO_2 and H_2O
c) reactants PbO_2 and HCl; products $PbCl_2$, Cl_2 and H_2O
d) reactants C_2H_5OH and O_2; products CO_2 and H_2O
e) reactants MnO_2, H_2SO_4 and $H_2C_2O_4$; products $MnSO_4$, CO_2 and H_2O

20. Two particles, of mass m_1 and m_2 (kilograms), are arranged with light ropes and smooth light pulleys as shown in the diagram below:



a) Consider first of all the case when $m_1 = m_2 = 3$.
(i) Show that the augmented matrix for a system of linear equations in the three variables $\ddot{x}_1, \ddot{x}_2, T$, where T denotes the tension of the rope, has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -g/5 \\ 0 & 1 & 0 & 2g/5 \\ 0 & 0 & 1 & 9g/5 \end{array}\right).$$

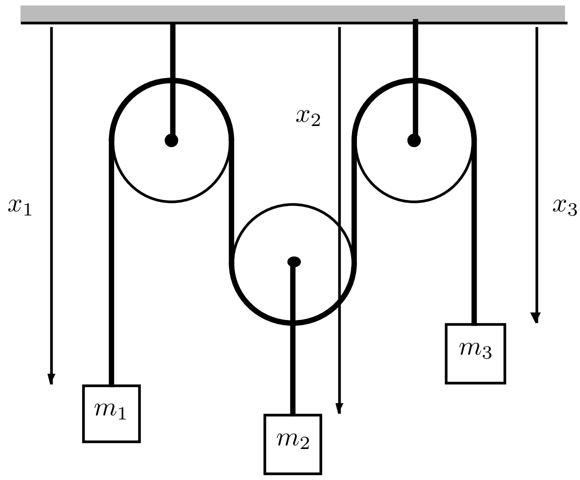
(ii) Determine $\ddot{x}_1, \ddot{x}_2, T$ in this case.
(iii) In which direction is the particle on the right moving?

b) Show that in the general case, we have

$$(\ddot{x}_1, \ddot{x}_2, T) = \left(\frac{(m_1 - 2m_2)g}{m_1 + 4m_2}, \frac{2(2m_2 - m_1)g}{m_1 + 4m_2}, \frac{3m_1 m_2 g}{m_1 + 4m_2} \right).$$

c) What relationship must m_1 and m_2 have in order to achieve equilibrium?

21. Three particles, of mass m_1, m_2 and m_3 (kilograms), are arranged with light ropes and smooth light pulleys as shown in the diagram below:



a) Show that we have

$$(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3, T) = \left(\left(1 - \frac{4m_2 m_3}{M}\right) g, \left(1 - \frac{8m_1 m_3}{M}\right) g, \left(1 - \frac{4m_1 m_2}{M}\right) g, \frac{4m_1 m_2 m_3 g}{M} \right),$$

where T denotes the tension of the rope and $M = m_1 m_2 + m_2 m_3 + 4m_1 m_3$.

b) Show that equilibrium occurs precisely when $m_2 = 2m_1 = 2m_3$.