First Order Linear Systems with Constant Coefficients

6.1. Homogeneous Systems

Consider the first order homogeneous linear system

\[ x' = Ax. \]  

Here \( A \) is a constant \( n \times n \) matrix with real or complex entries, and \( x = x(t) = (x_1(t), \ldots, x_n(t)) \) is an unknown vector function, considered as an \( n \)-dimensional column vector.

It was shown in Chapter 3 that given any initial condition

\[ x(t_0) = x_0 = (x_{10}, \ldots, x_{n0}), \]  

there is a unique solution to the system (6.1) satisfying (6.2). It was also shown that a fundamental system of solutions \( \varphi_1(t), \ldots, \varphi_n(t) \) of (6.1) exists. Here in the special case when \( A \) is constant, we shall attempt to describe such a fundamental system.

Suppose that \( B \) is a constant invertible \( n \times n \) matrix. Write \( y = B^{-1}x \). Then \( x = By \). Furthermore, \( x' = By' \). It follows that (6.1) can be described by \( By' = ABy \), i.e.

\[ y' = B^{-1}ABy. \]

The idea is to choose \( B \) so that \( B^{-1}AB \) has simple form.

Consider the polynomial \( \det(A - pI) = 0 \). This can be written in the form

\[ (p - \lambda_1)^{\mu_1} \cdots (p - \lambda_k)^{\mu_k} = 0, \]

where the distinct roots \( \lambda_1, \ldots, \lambda_k \in \mathbb{C} \) are the eigenvalues, with multiplicities \( \mu_1, \ldots, \mu_k \) respectively. It is well known that we can choose \( B \) so that \( B^{-1}AB \) is in Jordan normal form. When this is achieved, we can then study the system (6.3) in Section 6.2. The reduction of the matrix \( A \) to Jordan normal form is covered in any standard course in linear algebra. However, for the sake of completeness, we briefly discuss this in Section 6.3.

6.2. A Fundamental System of Solutions

Suppose now that we have found an invertible matrix \( B \) such that \( B^{-1}AB \) is in Jordan normal form; in other words,

\[ J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{pmatrix}, \]
where, for every \( j = 1, \ldots, k \), the \( \mu_j \times \mu_j \) matrix

\[
J_j = \begin{pmatrix}
\lambda_j & v_2 & \cdots & v_{\mu_j} \\
v_2 & \lambda_j & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
v_{\mu_j} & \cdots & \cdots & \lambda_j \\
\end{pmatrix},
\]

with \( v_2, \ldots, v_{\mu_j} \in \{0, 1\} \), and \( \lambda_1, \ldots, \lambda_k \) are distinct eigenvalues of the matrix \( A \), with multiplicities \( \mu_1, \ldots, \mu_k \) respectively. This means that

\[
(6.5)
J = \begin{pmatrix}
J_1^* \\
J_2^* \\
\vdots \\
J_m^* \\
\end{pmatrix},
\]

where for every \( k = 1, \ldots, m \), the matrix \( J_k^* \) is of the form

\[
(6.6)
\begin{pmatrix}
\lambda & 1 \\
1 & \lambda \\
\vdots & \vdots \\
\vdots & \vdots \\
1 & \lambda \\
\end{pmatrix}
\]

for some eigenvalue \( \lambda \) of \( A \) and some \( \mu \leq n \). Note that the numbers \( \lambda \) may no longer be distinct.

Consider first of all the first order \( n \)-dimensional system

\[
(6.7) \quad y' = Ky,
\]

where the matrix \( K \) is of the form (6.6).

**Lemma 6.1.** A fundamental system of solutions of the first order \( \mu \)-dimensional system (6.7) is given by

\[
\phi_1[\lambda, \mu][t] = (e^{\lambda t}, 0, \ldots, 0),
\]

\[
\phi_2[\lambda, \mu][t] = (te^{\lambda t}, e^{\lambda t}, 0, \ldots, 0),
\]

\[
\phi_3[\lambda, \mu][t] = \left( \frac{1}{2!} t^2 e^{\lambda t}, te^{\lambda t}, e^{\lambda t}, 0, \ldots, 0 \right),
\]

\[
\phi_4[\lambda, \mu][t] = \left( \frac{1}{3!} t^3 e^{\lambda t}, \frac{1}{2!} t^2 e^{\lambda t}, te^{\lambda t}, e^{\lambda t}, 0, \ldots, 0 \right),
\]

\[
\vdots
\]

\[
\phi_\mu[\lambda, \mu][t] = \left( \frac{1}{(\mu - 1)!} t^{\mu-1} e^{\lambda t}, \ldots, \frac{1}{2!} t^2 e^{\lambda t}, te^{\lambda t}, e^{\lambda t} \right).
\]

**Proof.** It is not difficult to check for every \( j = 1, \ldots, \mu \), the vector \( \phi_j[\lambda, \mu][t] \) is a solution of the system (6.7). Note also that the Wronskian \( W(0) = 1 \). \( \Box \)

Consider now of all the first order \( n \)-dimensional system

\[
(6.8) \quad y' = Jy,
\]

where the matrix \( J \) is of the form (6.5), and where, for every \( k = 1, \ldots, m \), the matrix \( J_k^* \) is of the form (6.6) for some eigenvalue \( \lambda_k \) of \( A \) and some \( \mu_k \leq n \).
Lemma 6.2. A fundamental system of solutions of the first order \( \mu \)-dimensional system (6.8) is given by

\[
\begin{align*}
(\phi_1[\lambda_1, \mu_1](t), & 0, \ldots, 0)_{\mu_2 + \ldots + \mu_m} \\
& \vdots \\
(\phi_{\mu_1}[\lambda_1, \mu_1](t), & 0, \ldots, 0)_{\mu_2 + \ldots + \mu_m} \\
& \vdots \\
(0, & \ldots, 0, \phi_1[\lambda_2, \mu_2](t), 0, \ldots, 0)_{\mu_3 + \ldots + \mu_m} \\
& \vdots \\
(0, & \ldots, 0, \phi_{\mu_2}[\lambda_2, \mu_2](t), 0, \ldots, 0)_{\mu_3 + \ldots + \mu_m} \\
& \vdots \\
(0, & \ldots, 0, \phi_1[\lambda_3, \mu_3](t))_{\mu_4 + \ldots + \mu_m} \\
& \vdots \\
(0, & \ldots, 0, \phi_{\mu_3}[\lambda_3, \mu_3](t)).
\end{align*}
\]

Proof. It is not difficult to check for every vector given is a solution of the system (6.8), in view of Lemma 6.1. Note again that the Wronskian \( W(0) = 1 \).

We now return to the system (6.1).

Lemma 6.3. Suppose that \( \psi_1(t), \ldots, \psi_n(t) \) form a fundamental system of solutions of the system (6.8), where \( J = B^{-1}AB \) is in Jordan normal form. Then \( B\psi_1(t), \ldots, B\psi_n(t) \) form a fundamental system of solutions of the system (6.1).

Proof. For every \( k = 1, \ldots, n \), since \( \psi_k'(t) = J\psi_k(t) \), it follows that \( B\psi_k'(t) = BJB^{-1}B\psi_k(t) \). Note now that \( A = BJB^{-1} \). On the other hand, the Wronskian of \( B\psi_1(t), \ldots, B\psi_n(t) \) is the product of the determinant of \( B \) and the Wronskian of \( \psi_1(t), \ldots, \psi_n(t) \), and so non-zero.

To summarize, given any first order homogeneous linear system \( x' = Ax \), we first reduce the matrix \( A \) to Jordan normal form \( J \). We then find a fundamental system of solutions of the system \( y' = Jy \) by using Lemma 6.2. Finally, we obtain a fundamental system of solutions of the original system in view of Lemma 6.3.

Example. Consider the 3-dimensional system \( x' = Ax \), where

\[
A = \begin{pmatrix}
-3 & 1 & -1 \\
-7 & 5 & -1 \\
-6 & 6 & -2
\end{pmatrix}.
\]

If

\[
B = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & -1 & 1
\end{pmatrix},
\]

then it is not difficult to check that

\[
B^{-1}AB = J = \begin{pmatrix}
-2 & 1 & 0 \\
0 & -2 & 0 \\
0 & 0 & 4
\end{pmatrix}.
\]
We now need to solve the 3-dimensional system \( y' = Jy \). By Lemma 6.2, a fundamental system of solutions of \( y' = Jy \) is given by
\[
\psi_1(t) = (e^{-2t}, 0, 0), \\
\psi_2(t) = (te^{-2t}, e^{-2t}, 0), \\
\psi_3(t) = (0, 0, e^{4t}).
\]
By Lemma 6.3, a fundamental system of solutions of \( x' = Ax \) is given by
\[
B\psi_1(t) = (e^{-2t}, e^{-2t}, 0), \\
B\psi_2(t) = (te^{-2t} + e^{-2t}, te^{-2t} + e^{-2t}, -e^{-2t}), \\
B\psi_3(t) = (0, e^{4t}, e^{4t}).
\]

### 6.3. Jordan Normal Form

This section is concerned with describing the linear algebra necessary to reduce the matrix \( A \) to Jordan normal form, and may be skipped if you are familiar with Jordan normal form.

**Lemma 6.4.** Let \( A \) be an \( n \times n \) matrix with characteristic polynomial (6.4). For every \( j = 1, \ldots, k \), let \( V_j = \{ u \in \mathbb{C}^n : (A - \lambda_j I)^{\mu_j} u = 0 \} \). Then \( \mathbb{C}^n \) is the direct sum of \( V_1, \ldots, V_k \).

**Proof.** (i) We shall first of all show that \( \mathbb{C}^n \) is the direct sum of \( V_1 \) and \( W \), where
\[
W = \{ u \in \mathbb{C}^n : (A - \lambda_2 I)^{\mu_2} \cdots (A - \lambda_k I)^{\mu_k} u = 0 \}.
\]
To do this, let
\[
f_1(p) = (p - \lambda_1)^{\mu_1} \quad \text{and} \quad f_2(p) = (p - \lambda_2)^{\mu_2} \cdots (p - \lambda_k)^{\mu_k}.
\]
Then \( f_1(p) \) and \( f_2(p) \) are coprime, so there exist polynomials \( g_1(p) \) and \( g_2(p) \), with coefficients in \( \mathbb{C} \), such that \( g_1(p)f_1(p) + g_2(p)f_2(p) = 1 \). Hence
\[
g_1(A)f_1(A) + g_2(A)f_2(A) = I.
\]
Let \( u \in \mathbb{C}^n \). Then
\[
u = g_1(A)f_1(A)u + g_2(A)f_2(A)u.
\]
Now \( g_1(A)f_1(A)u \in W \), since
\[
f_2(A)g_1(A)f_1(A)u = g_1(A)f_1(A)f_2(A)u = 0.
\]
Similarly \( g_2(A)f_2(A)u \in V_1 \). Hence \( \mathbb{C}^n \) is a sum of \( V_1 \) and \( W \). To show that \( \mathbb{C}^n \) is the direct sum of \( V_1 \) and \( W \), we shall show that for every \( u \in \mathbb{C}^n \), the expression
\[
u = v_1 + w, \quad v_1 \in V, \quad w \in W,
\]
is unique. Note first of all that since \( f_1(A)v_1 = 0 \), we must have
\[
g_1(A)f_1(A)u = g_1(A)f_1(A)w.
\]
On the other hand, using (6.9) and noting that \( f_2(A)w = 0 \), we must have
\[
w = g_1(A)f_1(A)w.
\]
Combining (6.10) and (6.11), we conclude that \( w = g_1(A)f_1(A)u \). Similarly \( v_1 = g_2(A)f_2(A)u \).

(ii) We now assume that \( W \) is a direct sum of \( V_2, \ldots, V_k \), where for every \( j = 2, \ldots, k \),
\[
V_j' = \{ u \in W : (A - \lambda_j I)^{\mu_j} u = 0 \}.
\]
Clearly \( V_j' \subseteq V_j \). On the other hand, suppose that \( u \in V_j \). Then \( (A - \lambda_j I)^{\mu_j} u = 0 \), so that \( (A - \lambda_2 I)^{\mu_2} \cdots (A - \lambda_k I)^{\mu_k} u = 0 \), whence \( u \in W \). Hence \( V_j \subseteq V_j' \). \( \Box \)
6.3. JORDAN NORMAL FORM

Lemma 6.5. Let \( V \) be a vector space over \( \mathbb{C} \), of dimension \( \mu \). Suppose that \( f : V \to V \) is a linear map, and that there exists an integer \( r \geq 0 \) such that \( f^r = 0 \) on \( V \). Then there exists a basis of \( V \) with respect to which the matrix of \( f \) is of the form

\[
\begin{pmatrix}
0 & v_2 & v_3 \\
v_2 & 0 & \ddots \\
& \ddots & \ddots \\
& & \ddots & 0
\end{pmatrix},
\]

where \( v_2, \ldots, v_\mu \in \{0, 1\} \).

To prove Lemma 6.5, we may assume that \( f \neq 0 \) on \( V \), otherwise the result is trivial. Hence there exists an integer \( q \geq 1 \) such that

\[ f^q \neq 0 \quad \text{and} \quad f^{q+1} = 0 \]
on \( V \). For every integer \( r \geq 0 \), let

\[ E_r = \ker f^r. \]

Clearly \( E_0 = \{0\} \) and \( E_{q+1} = V \).

Lemma 6.6. We have \( \{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_q \subsetneq E_{q+1} = V \). Furthermore, \( f(E_{i+1}) \subseteq E_i \) for every \( i = 0, 1, \ldots, q \).

Lemma 6.7. Let \( i = 1, \ldots, q \) be chosen. Suppose that \( W \) is a subspace of \( V \) such that \( W \cap E_i = \{0\} \). Then \( f(W) \cap E_{i-1} = \{0\} \). Furthermore, \( f \) induces an isomorphism of \( W \) onto \( f(W) \).

Lemma 6.8. There exist subspaces \( W_1, \ldots, W_{q+1} \) of \( V \) such that

(i) for every \( i = 1, \ldots, q+1 \), \( E_i \) is the direct sum of \( E_{i-1} \) and \( W_i \); and
(ii) for every \( i = 2, \ldots, q+1 \), \( f \) maps \( W_i \) one-to-one into \( W_{i-1} \).

Proof of Lemma 6.5. We first of all construct the subspaces \( W_1, \ldots, W_{q+1} \) of \( V \) as in Lemma 6.8. Let

\[ v_{1,1}, v_{1,2}, \ldots, v_{1,r_1} \]

be a basis of \( W_{q+1} \). Since these vectors are linearly independent and since \( f \) maps \( W_{q+1} \) one-to-one into \( W_q \), the vectors

\[ f(v_{1,1}), f(v_{1,2}), \ldots, f(v_{1,r_1}) \]

are linearly independent. We can extend this collection to a basis of \( W_q \) of the form

\[ v_{2,1}, \ldots, v_{2,r_1}, v_{2,r_1+1}, \ldots, v_{2,r_2}, \]

where

\[ f(v_{1,j}) = v_{2,j}, \quad j = 1, \ldots, r_1. \]

Repeating this argument, we can show that there exists a basis of \( W_{q-1} \) of the form

\[ v_{3,1}, \ldots, v_{3,r_2}, v_{3,r_2+1}, \ldots, v_{3,r_3}, \]

where

\[ f(v_{2,j}) = v_{3,j}, \quad j = 1, \ldots, r_2. \]

Continuing in this way, we finally arrive at a basis of \( W_1 = E_1 \) of the form

\[ (6.12) \quad v_{q+1,1}, \ldots, v_{q+1,r_{q+1}}, \]

where

\[ f(v_{q+1,j}) = v_{q+1,j}, \quad j = 1, \ldots, r_q. \]

Since \( E_1 = \ker f \), we also have

\[ f(v_{q+1,j}) = 0, \quad j = 1, \ldots, r_{q+1}. \]
It is easy to see from Lemma 6.8(i) that $V$ is the direct sum of $W_1, \ldots, W_{q+1}$, and a basis can be constructed using the vectors we have discussed so far. We can write the elements of this basis in the following order:

\[
\begin{array}{cccc}
  v_{1,1} & \cdots & v_{1,r_1} \\
  v_{2,1} & \cdots & v_{2,r_1} & v_{2,r_1+1} & \cdots & v_{2,r_2} \\
  \vdots & & \vdots & \vdots & \vdots \\
  v_{q+1,1} & \cdots & v_{q+1,r_1} & v_{q+1,r_1+1} & \cdots & v_{q+1,r_2} & \cdots & v_{q+1,r_{q+1}+1} & \cdots & v_{q+1,r_{q+1}} \\
\end{array}
\]

The elements can be rewritten in the following order: column by column from left to right, and each column from bottom to top, and denoted by

\[
v_i, \quad i = 1, \ldots, r_1 + \ldots + r_{q+1}.
\]

Then either $f(v_i) = 0$ or $f(v_i) = v_{i-1}$, and the matrix of $f$ with respect to this basis has the form indicated in the statement of Lemma 6.5.  

**Proof of Lemma 6.6.** The second assertion follows from the definition of $E_i$. To prove the first assertion, note that clearly $E_i \subseteq E_{i+1}$. On the other hand, suppose on the contrary that $E_i = E_{i+1}$ for some $i = 0, 1, \ldots, q$. Then for every $v \in V$, $0 = f^{q+1}(v) = f^{i+1}(f^{q-i}(v))$, so that $f^{q-i}(v) \in E_{i+1} = E_i$, whence $f^i(v) = 0$, contradicting the definition of $q$.  

**Proof of Lemma 6.7.** Let $v \in f(W) \cap E_{i-1}$. Then there exists $w \in W$ such that $v = f(w)$, so that $0 = f^{q-i}(v) = f^i(w)$, whence $w \in E_i$. Hence $w = 0$, and so $v = 0$. This proves the first assertion. On the other hand, the mapping $f|_W : W \to f(W)$ is clearly linear and onto. It is easy to check that it is also one-to-one.  

**Proof of Lemma 6.8.** Choose $W_{q+1}$ so that $E_{q+1} = V$ is a direct sum of $E_q$ and $W_{q+1}$. Then $f(W_{q+1}) \subseteq E_q$. By Lemma 6.7, $f(W_{q+1}) \cap E_{q-1} = \{0\}$. Hence there exists $W_q$ such that $f(W_{q+1}) \subseteq W_q$ and such that $E_q$ is a direct sum of $E_{q-1}$ and $W_q$. We continue this way to construct $W_{q-1}, \ldots, W_1$, where $f(W_{i+1}) \subseteq W_i$ for every $i = 1, \ldots, q$. Note that (ii) follows from the second assertion of Lemma 6.7.  

To obtain Jordan normal form, we first of all apply Lemma 6.4. Consider the restriction of $A$ to the elements of each $V_j$. Then $A - \lambda_i I$ can be described by a matrix of the form given in Lemma 6.5. Obtain a Jordan bases for $V_j$. We now take the union of all such bases obtained for $V_1, \ldots, V_k$. Jordan normal form follows easily.