8.1. Removable Singularities

Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \). Observe that it is not necessary for \( f \) to be defined at the point \( z_0 \). We say that the function \( f \) has an isolated singularity at \( z_0 \).

Our purpose is to show that there are only three possible ways in which \( f(z) \) can behave in a punctured neighbourhood of \( z_0 \). To illustrate the first of these, let us first consider the following examples.

**Example 8.1.1.** The function

\[
 f(z) = \frac{\sin z}{z}
\]

is analytic in the punctured disc \( \{ z : 0 < |z| < R \} \). However, the quotient is not defined at \( z = 0 \). However, note that the function \( \sin z \) is entire. By Theorem 7C, we can write

\[
 \sin z = z + z^3 g(z),
\]

where \( g \) is an entire function. It follows that for \( z \neq 0 \), we have

\[
 f(z) = \frac{\sin z}{z} = 1 + z^2 g(z).
\]

Note that the function \( 1 + z^2 g(z) \) is entire. It follows that if we make the further definition \( f(0) = 1 \), then \( f \) is now analytic at \( z = 0 \), and we have removed the isolated singularity.
Example 8.1.2. Suppose that a function \( f \) is analytic in a domain \( D \), and that \( z_0 \in D \). We define the function \( g \) in \( D \) by writing
\[
g(z_0) = f'(z_0),
\]
and writing
\[
g(z) = \frac{f(z) - f(z_0)}{z - z_0}
\]
if \( z \neq z_0 \). It is easily seen from Theorem 7C that \( g \) is analytic in \( D \). However, note that the function \( g \), defined by (2), is analytic in the domain \( D \setminus \{z_0\} \). It also has an isolated singularity at \( z_0 \), which is removed by the definition (1).

Definition. Suppose that a function \( f \) is analytic in the punctured disc \( \{z : 0 < |z - z_0| < R\} \). Suppose further that by assigning a suitable value for \( f(z_0) \), the function \( f \) can be made to be analytic in the disc \( \{z : |z - z_0| < R\} \). Then we say that \( f \) has a removable singularity at \( z_0 \).

Theorem 8A. (Riemann’s Theorem on Removable Singularities) Suppose that a function \( f \) is analytic in the punctured disc \( \{z : 0 < |z - z_0| < R\} \). Suppose further that
\[
\lim_{z \to z_0} (z - z_0)f(z) = 0.
\]
Then \( f \) has a removable singularity at \( z_0 \).

Proof. Suppose that \( z \) is a point in the punctured disc \( \{z : 0 < |z - z_0| < R\} \). Then \( 0 < |z - z_0| < R \).

Let \( r_1 \) and \( r_2 \) satisfy \( 0 < r_1 < |z - z_0| < r_2 < R \), and let \( C_1 \) and \( C_2 \) denote two circles in the positive (anticlockwise) direction, centred at \( z_0 \), and of radius \( r_1 \) and \( r_2 \) respectively.

The function \( g \), defined by \( g(z) = f'(z) \) and for \( \zeta \neq z \) by
\[
g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z},
\]
is clearly analytic in the punctured disk \( \{\zeta : 0 < |\zeta - z_0| < R\} \). Then it can be shown, as in the proof of Theorem 6A, that
\[
\int_{C_1} g(\zeta) \, d\zeta = \int_{C_2} g(\zeta) \, d\zeta.
\]
Combining this with (4), we have
\[
\int_{C_1} f(\zeta) \frac{d\zeta}{\zeta - z} \bigg|_{\zeta} - f(z) \int_{C_2} \frac{d\zeta}{\zeta - z} = \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) \int_{C_2} \frac{d\zeta}{\zeta - z}. \tag{5}
\]

Note now that the function
\[
\frac{1}{\zeta - z}
\]
is analytic in the star domain \( \{ \zeta : |\zeta - z_0| < |z - z_0| \} \) which contains the contour \( C_1 \). It follows that
\[
\int_{C_1} \frac{d\zeta}{\zeta - z} = 0. \tag{6}
\]

On the other hand, by Cauchy’s integral formula as given by Theorem 6A, we have
\[
\int_{C_2} \frac{d\zeta}{\zeta - z} = 2\pi i. \tag{7}
\]

Furthermore, in view of the condition (3), we have, given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |(\zeta - z_0)f(\zeta)| < \epsilon \) whenever \( |\zeta - z_0| < \delta \). Without loss of generality, we may assume that
\[
\delta < \frac{1}{2}|z - z_0|. \tag{8}
\]

If we now take \( r_1 = \delta \), then
\[
\left| \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \right| = \left| \int_{C_1} \frac{(\zeta - z_0)f(\zeta)}{(\zeta - z_0)(\zeta - z)} d\zeta \right| \leq \frac{\epsilon}{\delta(|z - z_0| - \delta)} 2\pi\delta = \frac{2\pi\epsilon}{|z - z_0| - \delta} \leq \frac{4\pi\epsilon}{|z - z_0|},
\]
in view of Theorem 4B and (8). Since \( \epsilon > 0 \) is arbitrary, we conclude that
\[
\int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta = 0. \tag{9}
\]

Combining (5)–(7) and (9), we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{10}
\]

Note now that (10) holds for every \( z \) in the punctured disc \( \{ z : 0 < |z - z_0| < r_2 \} \). Note also that the integral on the right hand side of (10) represents an analytic function in the disc \( \{ z : |z - z_0| < r_2 \} \) (see the proof of Theorem 6B). It follows that if we define
\[
f(z_0) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z_0} d\zeta,
\]

then the function \( f \) is analytic in the disc \( \{ z : |z - z_0| < r_2 \} \).

Remarks. (1) Note that condition (3) will be satisfied if \( f(z) \) is continuous at \( z_0 \), or if \( |f(z)| \) is bounded.

(2) Since an analytic function is continuous, it follows that removable singularities at \( z_0 \) can be overcome by defining
\[
f(z_0) = \lim_{z \to z_0} f(z).
\]
8.2. Poles

DEFINITION. Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \). Suppose further that

\[
f(z) = \frac{g(z)}{(z - z_0)^n},
\]

where \( n \in \mathbb{N} \) and the function \( g \) is analytic in some neighbourhood of \( z_0 \), with \( g(z_0) \neq 0 \). Then we say that \( f \) has a pole of order \( n \) at \( z_0 \). Furthermore, if \( n = 1 \), then we say that \( f \) has a simple pole at \( z_0 \).

THEOREM 8B. Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \). Then \( f \) has a pole at \( z_0 \) if and only if

\[
\lim_{z \to z_0} |f(z)| = \infty;
\]

in other words, given any \( E > 0 \), there exists \( \delta > 0 \) such that \( |f(z)| > E \) whenever \( 0 < |z - z_0| < \delta \).

PROOF. Note first of all that (12) follows immediately from (11), since \( g(z_0) \neq 0 \). Suppose now that (12) holds. Then \( f(z) \neq 0 \) in some punctured disc \( \{ z : 0 < |z - z_0| < r \} \), where \( r \leq R \). It follows that the function

\[
F(z) = \frac{1}{f(z)}
\]

is analytic in \( \{ z : 0 < |z - z_0| < r \} \), and has an isolated singularity at \( z_0 \). On the other hand, it follows from (12) that \( F(z) \to 0 \) as \( z \to z_0 \). Hence by Theorem 8A, \( F \) has a removable singularity at \( z_0 \). If we define \( F(z_0) = 0 \), then \( F \) is now analytic in the disc \( \{ z : |z - z_0| < r \} \). Clearly \( F(z) \) is not identically zero in \( \{ z : |z - z_0| < r \} \). It follows from Theorem 7F that there exists \( n \in \mathbb{N} \) such that

\[
F(z) = (z - z_0)^n h(z),
\]

where the function \( h \) is analytic in \( \{ z : |z - z_0| < r \} \), with \( h(z_0) \neq 0 \). Hence

\[
g(z) = \frac{1}{h(z)}
\]

is analytic in some neighbourhood of \( z_0 \), and (11) holds. Clearly \( g(z_0) \neq 0 \).

REMARK. Note that a function \( f \) has a pole of order \( n \) at \( z_0 \) if and only if the function \( 1/f \) has a zero of order \( n \) at \( z_0 \).

8.3. Essential Singularities

DEFINITION. Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \). Suppose further that the isolated singularity at \( z_0 \) is neither removable nor a pole. Then we say that \( f \) has an essential singularity at \( z_0 \).

EXAMPLE 8.3.1. The function \( e^{1/z} \) is analytic at every \( z \neq 0 \). It has an isolated singularity at \( z = 0 \). Let us restrict \( z \) to be real numbers, and consider \( e^{1/x} \), where \( x > 0 \). Clearly

\[
\lim_{x \to 0^+} e^{1/x} = \lim_{y \to +\infty} e^y = \infty,
\]
so that the singularity is not removable. On the other hand, for every \( n \in \mathbb{N} \),

\[
\lim_{x \to 0^+} x^n e^{1/x} = \lim_{y \to +\infty} \frac{e^y}{y^n} = \infty,
\]

so that the singularity is not a pole of order \( n \). Hence \( e^{1/z} \) has an essential singularity at \( z = 0 \).

To illustrate the wild behaviour of an analytic function near an essential singularity, we mention Picard’s theorem that such a function assumes all values except possibly one in any neighbourhood of an essential singularity. The following result is somewhat weaker, and shows that such a function comes arbitrarily close to any given complex number in any neighbourhood of an essential singularity.

**THEOREM 8C. (CASORATI-WEIERSTRASS)** Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \), with an essential singularity at \( z_0 \). Then given any \( w \in \mathbb{C} \) and any real numbers \( \epsilon > 0 \) and \( \delta > 0 \), there exists \( z \) in the punctured disc satisfying

\[
0 < |z - z_0| < \delta \quad \text{and} \quad |f(z) - w| < \epsilon.
\]

**Proof.** Suppose on the contrary that the conclusion does not hold. Then there exist \( w \in \mathbb{C} \) and real numbers \( \epsilon > 0 \) and \( \delta > 0 \) such that \( |f(z) - w| \geq \epsilon \) whenever \( 0 < |z - z_0| < \delta \). It follows that the function

\[
g(z) = \frac{1}{f(z) - w}
\]

is analytic and bounded in the punctured disc \( \{ z : 0 < |z - z_0| < \delta \} \), with an isolated singularity at \( z_0 \) which is removable, in view of Theorem 8A. It follows that by defining \( g(z_0) \) appropriately, the function \( g \) is analytic in the disc \( \{ z : |z - z_0| < \delta \} \). On the other hand, the function \( g \) is clearly not identically zero in \( \{ z : |z - z_0| < \delta \} \). Furthermore, note that

\[
f(z) = w + \frac{1}{g(z)}.
\]

If \( g(z_0) \neq 0 \), then \( f \) is analytic at \( z_0 \). If \( g(z_0) = 0 \), then \( f \) has a pole at \( z_0 \). In either case, the conclusion contradicts the assumption that \( f \) has an essential singularity at \( z_0 \), and this completes the proof. \( \square \)

### 8.4. Isolated Singularities at Infinity

The behaviour of a function \( f(z) \) at \( z = \infty \) can be studied via the behaviour of the function \( f(1/\zeta) \) at \( \zeta = 0 \). A punctured neighbourhood \( \{ \zeta : 0 < |\zeta| < R^{-1} \} \) of 0 then plays the same role as the “punctured” neighbourhood \( \{ z : R < |z| < \infty \} \) of \( \infty \).

Suppose now that a function \( f(z) \) is analytic in the domain \( \{ z : R < |z| < \infty \} \). Then by using \( z = 1/\zeta \) and considering \( \zeta = 0 \), we see that the function \( f(z) \) has an isolated singularity at \( z = \infty \). This may be a removable singularity, a pole or an essential singularity.

Corresponding to Theorem 8A, suppose that \( |f(z)/z| \to 0 \) as \( |z| \to \infty \). Then the singularity is removable by defining \( f(\infty) \) suitably to make \( f(z) \) continuous at \( z = \infty \). In other words, we need to define

\[
f(\infty) = \lim_{\zeta \to 0} f\left( \frac{1}{\zeta} \right).
\]
In the special case that $f(\infty) = 0$, then we say that $f$ has a zero at $z = \infty$. Furthermore, if $f$ is not identically zero, then, corresponding to Theorem 7F, there exists $n \in \mathbb{N}$ such that

$$f(z) = \frac{h(z)}{z^n},$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that $f$ has a zero of order $n$ at $z = \infty$.

Corresponding to Theorem 8B, suppose that $|f(z)| \to \infty$ as $|z| \to \infty$. Then $f$ has a pole at $z = \infty$, and there exists $n \in \mathbb{N}$ such that

$$f(z) = z^n h(z),$$

where $h(z)$ is analytic in $\{z : R < |z| < \infty\}$, and $h(\infty) \neq 0$. In this case, we say that $f$ has a pole of order $n$ at $z = \infty$.

Corresponding to Theorem 8C, suppose that the isolated singularity at $z = \infty$ is neither removable nor a pole. Then it is an essential singularity. In this case, given any $w \in \mathbb{C}$ and any real numbers $\epsilon > 0$ and $N > 0$, there exists $z$ in the domain $\{z : R < |z| < \infty\}$ satisfying

$$|z| > N \quad \text{and} \quad |f(z) - w| < \epsilon.$$ 

In other words, the function $f(z)$ comes arbitrarily close to any given complex number in any neighbourhood of $z = \infty$.

### 8.5. Further Examples

#### Example 8.5.1.

The function

$$f(z) = \frac{e^z - 1}{z(z - 1)}$$

is analytic at every $z \in \mathbb{C}$ except for isolated singularities at $z = 0, 1$. At $z = 1$, it has a simple pole; note that we can write

$$f(z) = \frac{g(z)}{z - 1} \quad \text{with} \quad g(z) = \frac{e^z - 1}{z},$$

and $g(1) \neq 0$. At $z = 0$, it has a removable singularity, since

$$\lim_{z \to 0} \frac{e^z - 1}{z(z - 1)} = \lim_{z \to 0} \frac{e^z}{2z - 1} = -1$$

by l’Hopital’s rule. It follows that if we define $f(0) = -1$, then $f$ is analytic at $z = 0$. The function $f(z)$ also has an isolated singularity at $z = \infty$. To study the isolated singularity at $z = \infty$, note first of all that

$$\lim_{|z| \to \infty} \frac{e^z}{z(z - 1)}$$

do not exist. To see this, note that

$$\lim_{x \to \infty} \frac{e^x - 1}{x(x - 1)} = +\infty \quad \text{and} \quad \lim_{x \to -\infty} \frac{e^x - 1}{x(x - 1)} = 0.$$
Hence the singularity is not removable. Suppose next that \( n \in \mathbb{N} \) is given and fixed. Then
\[
h(z) = \frac{f(z)}{z^n} = \frac{e^z - 1}{z^{n+1}(z - 1)}
\]
is not analytic at \( z = \infty \), since
\[
\lim_{|z|\to\infty} \frac{e^z - 1}{z^{n+1}(z - 1)}
\]
does not exist. To see this, note that
\[
\lim_{x \to +\infty} \frac{e^x - 1}{x^{n+1}(x - 1)} = +\infty \quad \text{and} \quad \lim_{x \to -\infty} \frac{e^x - 1}{x^{n+1}(x - 1)} = 0.
\]
Hence the singularity is not a pole. It follows that \( f(z) \) has an essential singularity at \( z = \infty \).

**Example 8.5.2.** The function
\[
f(z) = \frac{(z^2 - 4)(z - 1)^4}{(\sin \pi z)^4}
\]
is analytic at every \( z \in \mathbb{C} \) except for isolated singularities at \( z = 0, \pm 1, \pm 2, \ldots \), where the denominator vanishes. Note also that the numerator vanishes at \( z = 1, \pm 2 \). Note that the function \( \sin \pi z \) has simple zeros at \( z = 0, \pm 1, \pm 2, \ldots \). It follows that \( f \) has poles of order 4 at \( z = 0, -1, 1, 2 \). Next, note that the function \((z^2 - 4)(z - 1)^4\) has simple zeros at \( z = \pm 2 \). It follows that \( f \) has poles of order 3 at \( z = \pm 2 \). To study the isolated singularity at \( z = 1 \), note that by Theorem 7C, we have
\[
\sin \pi z = -\pi(z - 1) + g(z)(z - 1)^2,
\]
where \( g \) is entire. It follows that
\[
\lim_{z \to 1} \frac{(z^2 - 4)(z - 1)^4}{(\sin \pi z)^4} = \lim_{z \to 1} \frac{z^2 - 4}{(\pi - g(z)(z - 1))^4} = -\frac{3}{\pi^4},
\]
and so \( f \) has a removable singularity at \( z = 1 \). Finally, the singularity at \( z = \infty \) is not isolated, since there does not exist any \( R > 0 \) such that the function \( f(z) \) is analytic in the domain \( \{ z : R < |z| < \infty \} \).

### 8.6. Laurent Series

**Example 8.6.1.** Suppose that the function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \), with a pole of order \( m \) at \( z_0 \). Then
\[
f(z) = \frac{g(z)}{(z - z_0)^m},
\]
where the function \( g \) is analytic in \( \{ z : |z - z_0| < R \} \), with \( g(z_0) \neq 0 \). By Theorem 7C, we have
\[
g(z) = g(z_0) + g'(z_0)(z - z_0) + \frac{g''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{g^{(m-1)}(z_0)}{(m-1)!}(z - z_0)^{m-1} + g_m(z)(z - z_0)^m,
\]
where \( g_m(z) \) is analytic in the disc \( \{ z : |z - z_0| < R \} \). It follows that
\[
f(z) = \frac{g(z_0)}{(z - z_0)^m} + \frac{g'(z_0)}{(z - z_0)^{m-1}} + \frac{g''(z_0)}{2!(z - z_0)^{m-2}} + \ldots + \frac{g^{(m-1)}(z_0)}{(m-1)!(z - z_0)} + g_m(z).
\]
The expression
\[
g(z_0) \frac{g(z_0)}{(z - z_0)^m} + \frac{g'(z_0)}{(z - z_0)^{m-1}} + \frac{g''(z_0)}{2!(z - z_0)^{m-2}} + \cdots + \frac{g^{(m-1)}(z_0)}{(m-1)!(z - z_0)}
\]
is called the principal part of \( f \) at \( z_0 \). If we use Theorem 7A instead, then we can show that
\[
f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n
\]
for suitable choices of the coefficients \( a_n \).

Our main task in this section is to generalize this example. The first step in this direction can be summarized by the following result.

**THEOREM 8D.** Suppose that a function \( f \) is analytic in the punctured disc \( \{z : 0 < |z - z_0| < R\} \), with an isolated singularity at \( z_0 \). Then there exist unique functions \( f_1 \) and \( f_2 \) such that

(a) \( f(z) = f_1(z) + f_2(z) \) in \( \{z : 0 < |z - z_0| < R\} \),

(b) \( f_1 \) is analytic in \( \mathbb{C} \) except possibly at \( z_0 \),

(c) \( f_1(z) \to 0 \) as \( |z| \to \infty \), and

(d) \( f_2 \) is analytic in the disc \( \{z : |z - z_0| < R\} \).

**Proof.** We begin the proof in the same way as for Theorem 8A. Suppose that \( z \) is a point in the punctured disc \( \{z : 0 < |z - z_0| < R\} \). Let \( r_1 \) and \( r_2 \) satisfy \( 0 < r_1 < |z - z_0| < r_2 < R \), and let \( C_1 \) and \( C_2 \) denote two circles in the positive (anticlockwise) direction, centred at \( z_0 \), and of radius \( r_1 \) and \( r_2 \) respectively. On combining (5)–(7), we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]
(13)

Write
\[
f_1(z) = -\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]
(14)

Part (a) follows immediately. For part (d), note that the second integral in (14) represents an analytic function in the disc \( \{z : |z - z_0| < r_2\} \) (as in the proof of Theorems 6B and 8A). For part (b), note that the first integral in (14) represents an analytic function in the annulus \( \{z : |z - z_0| > r_1\} \) (similar to the proof of Theorem 6B). Note next that \( f_2(z) \) and \( f(z) \) are independent of the choice of \( r_1 \), so that it follows from (a) that \( f_1(z) \) is also independent of the choice of \( r_1 \). Similarly, \( f_2(z) \) is independent of the choice of \( r_2 \). It is easy to see that
\[
\lim_{|z| \to \infty} \int_{C_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta = 0.
\]
Part (c) follows immediately. To show that the functions \( f_1 \) and \( f_2 \) are unique, suppose that \( g_1 \) and \( g_2 \) are functions having the same properties as \( f_1 \) and \( f_2 \) respectively. Then
\[
f_1(z) - g_1(z) = g_2(z) - f_2(z)
\]
in the punctured disc \( \{z : 0 < |z - z_0| < R\} \). Let
\[
F(z) = \begin{cases} 
  g_2(z) - f_2(z) & \text{if } |z - z_0| < R, \\
  f_1(z) - g_1(z) & \text{if } |z - z_0| > 0.
\end{cases}
\]
Then $F$ is entire. On the other hand, it follows from part (c) that $F(z) \to 0$ as $|z| \to \infty$. Hence $F$ is bounded. It follows from Liouville’s theorem that $F$ is constant in $\mathbb{C}$, and so we must have $F(z) = 0$ for every $z \in \mathbb{C}$. This completes the proof. \(\square\)

We can now state our generalization of Example 8.6.1.

**Theorem 8E.** Suppose that a function $f$ is analytic in the punctured disc $\{z : 0 < |z - z_0| < R\}$, with an isolated singularity at $z_0$. For every $n \in \mathbb{Z}$, let

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz,$$

where $C$ is a circle in the positive (anticlockwise) direction centred at $z_0$ and of radius $r$, where $0 < r < R$. Then the series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is convergent in the punctured disc $\{z : 0 < |z - z_0| < R\}$. Furthermore, this convergence is uniform in any annulus $\{z : r_1 < |z - z_0| < r_2\}$, where $0 < r_1 < r_2 < R$.

**Remark.** To say that the series converges uniformly to $f(z)$ in the annulus $\{z : r_1 < |z - z_0| < r_2\}$, we mean given any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon, r_1, r_2)$, independent of the choice of $z$, such that

$$\left| f(z) - \sum_{n=-N_0}^{N_0} a_n (z - z_0)^n \right| < \epsilon$$

for every $z$ in the annulus $\{z : r_1 < |z - z_0| < r_2\}$ whenever $N_1 > N_0$ and $N_2 > N_0$.

**Definition.** The series (16) is called the Laurent series for the function $f$ at $z_0$.

**Proof of Theorem 8E.** The first step in our proof is to show that if the series in (16) converges uniformly on the circle $C$ centred at $z_0$ and of radius $r$, where $0 < r < R$, then the coefficients $a_n$ are given by (15). Suppose that $n \in \mathbb{Z}$ is chosen and fixed. For any $\epsilon > 0$, we can choose $N_1$ and $N_2$ so large that $-N_1 \leq n \leq N_2$ and

$$\left| f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right| < \epsilon$$

for every $z \in C$. Then it follows from Theorem 4B that

$$\left| \frac{1}{2\pi i} \int_C \left( f(z) - \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{dz}{(z - z_0)^{n+1}} \right| \leq \frac{\epsilon}{r^n}. \quad (17)$$

Since

$$\frac{1}{2\pi i} \int_C (z - z_0)^k \, dz = \begin{cases} 1 & \text{if } k = -1, \\ 0 & \text{if } k \neq -1, \end{cases}$$

we have

$$\frac{1}{2\pi i} \int_C \left( \sum_{j=-N_1}^{N_2} a_j (z - z_0)^j \right) \frac{dz}{(z - z_0)^{n+1}} = a_n.$$
so that (17) can be simplified to
\[ \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz - a_n \right| \leq \frac{\epsilon}{r^n}. \]

Since \( \epsilon > 0 \) is arbitrary, (15) follows immediately. It now remains to show that \( f(z) \) can be represented in the form (16) in the punctured disc \( \{ z : 0 < |z-z_0| < R \} \), and that the convergence is uniform in any annulus \( \{ z : r_1 < |z-z_0| < r_2 \} \), where \( 0 < r_1 < r_2 < R \). Suppose that \( 0 < r_1 < r < r_2 < R \). Following Theorem 8D, we can write
\[ f(z) = f_1(z) + f_2(z), \tag{18} \]
where \( f_1(z) \) and \( f_2(z) \) are uniquely determined and satisfy conditions (b)–(d) of Theorem 8D. Since \( f_2 \) is analytic in the disc \( \{ z : |z-z_0| < R \} \), it follows from Theorem 7A that the Taylor series
\[ f_2(z) = \sum_{n=0}^{\infty} A_n (z-z_0)^n \tag{19} \]
converges in the disc \( \{ z : |z-z_0| < R \} \), uniformly in the closed disc \( \{ z : |z-z_0| \leq r_2 \} \). To study \( f_1(z) \), write
\[ w = \frac{1}{z-z_0} \quad \text{or} \quad z = \frac{1}{w} + z_0. \]
Then
\[ f_1(z) = f_1 \left( \frac{1}{w} + z_0 \right) \]
is an entire function of \( w \), and so it follows from Theorem 7A that the Taylor series
\[ f_1 \left( \frac{1}{w} + z_0 \right) = \sum_{m=1}^{\infty} B_m w^m \tag{20} \]
converges in \( \mathbb{C} \), uniformly in the closed disc \( \{ w : |w| \leq 1/r_1 \} \). Note that the constant term \( B_0 \) in the Taylor series is missing, since \( B_0 \) corresponds to the value of the function at \( w = 0 \), or \( z = \infty \), and this is 0 in view of condition (c) in Theorem 8D. However, (20) is equivalent to saying that the series
\[ f_1(z) = \sum_{m=1}^{\infty} B_m (z-z_0)^{-m} \tag{21} \]
converges in \( \mathbb{C} \setminus \{0\} \), uniformly in \( \{ z : |z-z_0| \geq r_1 \} \). The result now follows on combining (18), (19) and (21).

**Definition.** The series
\[ f_1(z) = \sum_{n=-\infty}^{-1} a_n (z-z_0)^n, \]
where \( a_n \) is given by (15), is called the principal part of the function \( f \) at \( z_0 \).

The next result highlights the relationship between the principal part of a function and the nature of the isolated singularity.
THEOREM 8F. Suppose that a function \( f \) is analytic in the punctured disc \( \{ z : 0 < |z - z_0| < R \} \), with an isolated singularity at \( z_0 \). Suppose further that the Laurent coefficients \( a_n \) are given by (15).

(a) The function \( f \) either is analytic or has a removable singularity at \( z_0 \) if and only if \( a_n = 0 \) for every \( n < 0 \).

(b) The function \( f \) has a pole at \( z_0 \) if and only if a positive but finite number of coefficients \( a_n \) with \( n < 0 \) are non-zero.

(c) The function \( f \) has an essential singularity at \( z_0 \) if and only if an infinite number of coefficients \( a_n \) with \( n < 0 \) are non-zero.

Proof. Note first of all that if \( f \) has a removable singularity at \( z_0 \), then \( f \) can be made analytic at \( z_0 \) by a suitable choice of \( f(z_0) \). Part (a) now follows on observing that an analytic function has a Taylor series, and that a Laurent series with no principal part is a Taylor series. To prove part (b), note first of all that if a positive but finite number of coefficients \( a_n \) with \( n < 0 \) are non-zero, then there exists \( m > 0 \) such that \( a_{-m} \neq 0 \) but \( a_n = 0 \) for every \( n < -m \). In this case, we have

\[
f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n,
\]

so that

\[
f(z) = \frac{g(z)}{(z - z_0)^m},
\]

where \( m \in \mathbb{N} \) and the function \( g \) is analytic in some neighbourhood of \( z_0 \), with \( g(z_0) = a_{-m} \neq 0 \). This shows that \( f \) has a pole of order \( m \) at \( z_0 \). The converse is given in Example 8.6.1. Part (b) follows. Part (c) follows immediately from (a) and (b).

Example 8.6.2. The observation that a Laurent series is unique enables us to use different methods to find the coefficients apart from the formula (15). Consider, for example, the function \( e^{1/z} \). Using the substitution \( z = 1/w \) on the Taylor series

\[
e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!},
\]

we obtain the Laurent series

\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = \ldots + \frac{1}{3!z^3} + \frac{1}{2!z^2} + \frac{1}{z} + 1.
\]

We conclude this chapter by making a remark on various equivalent definitions of analyticity in a domain \( D \). The reader is advised to check the following theorem very carefully.

THEOREM 8G. For any function \( f \) and any domain \( D \), the following statements are equivalent:

(a) \( f(z) \) is analytic in \( D \).

(b) \( f(z) \) has continuous derivatives of all orders in \( D \).

(c) \( f'(z) \) exists and is continuous in \( D \).

(d) \( f'(z) \) exists in \( D \).

(e) \( f'(z) \) exists in \( D \) except possibly at a finite number of points in \( D \), and \( f(z) \) is continuous at these exceptional points.

(f) \( f(z) \) can be represented uniformly by its Taylor series in the neighbourhood of every point in \( D \).
PROBLEMS FOR CHAPTER 8

1. For each of the functions below, classify all the singular points in \( \mathbb{C} \):
   a) \( f(z) = e^z \)  
   b) \( f(z) = \frac{\cos z}{z} \)  
   c) \( f(z) = \frac{z^2 + 1}{z^2 - 1} \)  
   d) \( f(z) = \frac{z^4}{z^3 + z} \)  
   e) \( f(z) = -\frac{z}{\cos z} \)

2. Show that the principal parts of the function \( f(z) = 8z^3(z + 1)^{-1}(z - 1)^{-2} \) at \( z = -1 \) and \( z = 1 \) are respectively \(-2(z + 1)^{-1}\) and \(4(z - 1)^{-2} + 10(z - 1)^{-1}\).

3. For each of the functions below, find the principal part at the given points:
   a) \( f(z) = e^z \) at the point \( z = 0 \)  
   b) \( f(z) = \frac{z^6}{(1 - z)^3} \) at the point \( z = 1 \)  
   c) \( f(z) = \frac{\sin z}{(z - 2\pi)^2} \) at the point \( z = 2\pi \)

4. Expand the function \( (z - 1)/(z + 1) \) in powers of \( 1/z \).

5. For each of the functions below, use partial fractions if appropriate and find the principal part at each of its singular points in \( \mathbb{C} \):
   a) \( f(z) = \frac{12}{z^2(z^2 + 4)} \)  
   b) \( f(z) = \frac{z^4 + 1}{z(z^2 + 1)^2} \)  
   c) \( f(z) = \frac{48z^6}{(z - 1)^2(z - 2)} \)  
   d) \( f(z) = \frac{z^6 + 1}{(z - 1)^3(z^2 + 4)^2} \)

6. Suppose that \( f(z) = b_m z^{-m} + b_{m+1} z^{-m+1} + \ldots + b_0 + b_1 z + \ldots + b_k z^k \), where \( m, k \in \mathbb{N} \). Suppose further that \( f(z) \) has Laurent series
   \[
   \sum_{n=-\infty}^{\infty} a_n z^n
   \]
   at the point \( z = 0 \). Show by direct calculation that \( a_n = b_n \) whenever \(-m \leq n \leq k\) and \( a_n = 0 \) otherwise.

7. a) Consider the function \( f(z) = e^{1/z} \). Note that for every \( k \in \mathbb{Z} \), the coefficient for the term \( z^k \) in the Laurent series of \( f(z) \) at \( z = 0 \) is given by
   \[
a_k = \frac{1}{2\pi i} \int_C \frac{e^{1/\zeta}}{\zeta^{k+1}} d\zeta,
   \]
   where \( C \) is the circle \( \{ z : |z| = 1 \} \) followed in the positive (anticlockwise) direction. Show that
   \[
a_k = \frac{1}{2\pi} \int_0^{\pi} \frac{e^{\cos \theta} \cos(\sin \theta + k\theta)}{\cos(\sin \theta - n\theta)} d\theta = \frac{1}{n!}.
   \]
   b) Find the Laurent series for the function \( f(z) = e^{1/z} \) at \( z = 0 \) without using part (a).
   c) Deduce that for every \( n \in \mathbb{N} \cup \{0\} \),
   \[
   \frac{1}{n!} \int_0^{\pi} \frac{e^{\cos \theta} \cos(\sin \theta - n\theta)}{\cos(\sin \theta - n\theta)} d\theta = \frac{1}{n!}.
   \]