Chapter 1

THE NUMBER SYSTEM

1.1. The Real Numbers

The purpose of the first four sections of this chapter is to discuss a number of the properties of the real numbers. Most readers will be familiar with these properties, or have at least used most of them, perhaps sometimes unaware of their generality. We do not propose to discuss here these properties in great detail, and shall only give a brief introduction. Throughout, we denote the set of all real numbers by \( \mathbb{R} \), and write \( a \in \mathbb{R} \) to indicate that \( a \) is a real number.

The first collection of properties of \( \mathbb{R} \) is generally known as the Field axioms. We offer no proof of these properties, and simply treat and accept them as given.

FIELD AXIOMS.

(A1) For every \( a, b \in \mathbb{R} \), we have \( a + b \in \mathbb{R} \).

(A2) For every \( a, b, c \in \mathbb{R} \), we have \( a + (b + c) = (a + b) + c \).

(A3) For every \( a \in \mathbb{R} \), we have \( a + 0 = a \).

(A4) For every \( a \in \mathbb{R} \), there exists \( -a \in \mathbb{R} \) such that \( a + (-a) = 0 \).

(A5) For every \( a, b \in \mathbb{R} \), we have \( a + b = b + a \).

(M1) For every \( a, b \in \mathbb{R} \), we have \( ab \in \mathbb{R} \).

(M2) For every \( a, b, c \in \mathbb{R} \), we have \( a(bc) = (ab)c \).

(M3) For every \( a \in \mathbb{R} \), we have \( a1 = a \).

(M4) For every \( a \in \mathbb{R} \) such that \( a \neq 0 \), there exists \( a^{-1} \in \mathbb{R} \) such that \( aa^{-1} = 1 \).

(M5) For every \( a, b \in \mathbb{R} \), we have \( ab = ba \).

(D) For every \( a, b, c \in \mathbb{R} \), we have \( a(b + c) = ab + ac \).

Remark. The properties (A1)–(A5) concern the operation addition, while the properties (M1)–(M5) concern the operation multiplication. In the terminology of group theory, not usually covered in first
year mathematics, we say that the set $\mathbb{R}$ forms an abelian group under addition, and that the set of all non-zero real numbers forms an abelian group under multiplication. We also say that the set $\mathbb{R}$ forms a field under addition and multiplication. The property (D) is called the Distributive law.

The set of all real numbers also possesses an ordering relation, so we have the Order axioms.

**ORDER AXIOMS.**

(O1) For every $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$, $a > b$ holds.

(O2) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $b > c$, we have $a > c$.

(O3) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$, we have $a + c > b + c$.

(O4) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $c > 0$, we have $ac > bc$.

**Remark.** Clearly the Order axioms as given do not appear to include many other properties of the real numbers. However, these can be deduced from the Field axioms and Order axioms. For example, suppose that $x > 0$. Then by (A4), we have $-x \in \mathbb{R}$ and $x + (-x) = 0$. It follows from (O3) and (A3) that $0 = x + (-x) > 0 + (-x) = -x$, giving $-x < 0$.

### 1.2. The Natural Numbers

An important subset of the set $\mathbb{R}$ of all real numbers is the set of all natural numbers, usually given by $\mathbb{N} = \{1, 2, 3, \ldots\}$. However, this definition does not bring out some of the main properties of the set $\mathbb{N}$ in a natural way. The following more complicated definition is therefore sometimes preferred.

**Definition.** The set $\mathbb{N}$ of all natural numbers is defined by the following four conditions:

(N1) $1 \in \mathbb{N}$.

(N2) If $n \in \mathbb{N}$, then the number $n + 1$, called the successor of $n$, also belongs to $\mathbb{N}$.

(N3) Every $n \in \mathbb{N}$ other than 1 is the successor of some number in $\mathbb{N}$.

(WO) Every non-empty subset of $\mathbb{N}$ has a least element.

**Remark.** The condition (WO) is called the Well-ordering principle.

To explain the significance of each of these four requirements, note that the conditions (N1) and (N2) together imply that $\mathbb{N}$ contains $1, 2, 3, \ldots$. However, these two conditions alone are insufficient to exclude from $\mathbb{N}$ numbers such as 5.5. Now, if $\mathbb{N}$ contained 5.5, then by condition (N3), $\mathbb{N}$ must also contain 4.5, 3.5, 2.5, 1.5, 0.5, −0.5, −1.5, −2.5, ..., and so would not have a least element. We therefore exclude this possibility by stipulating that $\mathbb{N}$ has a least element. This is achieved by the condition (WO).

It can be shown that the condition (WO) implies the Principle of induction. The following two forms of the Principle of induction are particularly useful. In fact, both are equivalent to the condition (WO), as we shall show in Section 1.4.

**PRINCIPLE OF INDUCTION (WEAK FORM).** Suppose that the statement $p(.)$ satisfies the following conditions:

(PIW1) $p(1)$ is true; and

(PIW2) $p(n+1)$ is true whenever $p(n)$ is true.

Then $p(n)$ is true for every $n \in \mathbb{N}$.

**PRINCIPLE OF INDUCTION (STRONG FORM).** Suppose that the statement $p(.)$ satisfies the following conditions:

(PIS1) $p(1)$ is true; and

(PIS2) $p(n+1)$ is true whenever $p(m)$ is true for all $m \leq n$.

Then $p(n)$ is true for every $n \in \mathbb{N}$.
In the examples below, we shall illustrate some basic ideas involved in proof by induction.

**Example 1.2.1.** We shall prove by induction that

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]  

for every \( n \in \mathbb{N} \). To do so, let \( p(n) \) denote the statement (1). Then clearly \( p(1) \) is true. Suppose now that \( p(n) \) is true, so that

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \]

Then

\[ 1 + 2 + 3 + \ldots + n + (n + 1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}, \]

so that \( p(n+1) \) is true. It now follows from the Principle of induction (Weak form) that (1) holds for every \( n \in \mathbb{N} \).

**Example 1.2.2.** We shall prove by induction that

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \]  

for every \( n \in \mathbb{N} \). To do so, let \( p(n) \) denote the statement (2). Then clearly \( p(1) \) is true. Suppose now that \( p(n) \) is true, so that

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \]

Then

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \]
\[ = \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n + 2)(2n + 3)}{6}, \]

so that \( p(n+1) \) is true. It now follows from the Principle of induction (Weak form) that (2) holds for every \( n \in \mathbb{N} \).

**Example 1.2.3.** We shall prove by induction that \( 3^n > n^3 \) for every \( n > 3 \). To do so, let \( p(n) \) denote the statement

\[ (n \leq 3) \text{ or } (3^n > n^3). \]

Then clearly \( p(1), p(2), p(3), p(4) \) are all true. Suppose now that \( n > 3 \) and \( p(n) \) is true. Then \( 3^n > n^3 \). It follows that (note that we are aiming for \( (n+1)^3 = n^3 + 3n^2 + 3n + 1 \) all the way)

\[
3^{n+1} > 3n^3 = n^3 + 2n^3 > n^3 + 6n^2 = n^3 + 3n^2 + 3n^2 > n^3 + 3n^2 + 6n \\
= n^3 + 3n^2 + 3n + 3n > n^3 + 3n^2 + 3n + 1 = (n + 1)^3,
\]

so that \( p(n+1) \) is true. It now follows from the Principle of induction (Weak form) that \( 3^n > n^3 \) holds for every \( n > 3 \).
Example 1.2.4. We shall prove by induction the famous De Moivre theorem that
\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\] (3)
for every \(\theta \in \mathbb{R}\) and every \(n \in \mathbb{N}\). To do so, let \(\theta \in \mathbb{R}\) be fixed, and let \(p(n)\) denote the statement (3). Then clearly \(p(1)\) is true. Suppose now that \(p(n)\) is true, so that
\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.\]

Then
\[(\cos \theta + i \sin \theta)^{n+1} = (\cos n\theta + i \sin n\theta)(\cos \theta + i \sin \theta) = (\cos n\theta \cos \theta - \sin n\theta \sin \theta) + i(\sin n\theta \cos \theta + \cos n\theta \sin \theta) = \cos(n+1)\theta + i \sin(n+1)\theta,
so that \(p(n+1)\) is true. It now follows from the Principle of induction (Weak form) that (3) holds for every \(n \in \mathbb{N}\).

Example 1.2.5. Consider the sequence \(x_1, x_2, x_3, \ldots\), given by
\[x_1 = 5, \quad x_2 = 11\]
and \(x_{n+1} - 5x_n + 6x_{n-1} = 0 \quad \text{if} \quad n \geq 2.\) (4)
We shall prove by induction that
\[x_n = 2^{n+1} + 3^{n-1}\] (5)
for every \(n \in \mathbb{N}\). To do so, let \(p(n)\) denote the statement (5). Then clearly \(p(1), p(2)\) are both true. Suppose now that \(n \geq 2\) and \(p(m)\) is true for every \(m \leq n\), so that \(x_m = 2^{m+1} + 3^{m-1}\) for every \(m \leq n\). Then
\begin{align*}
x_{n+1} &= 5x_n - 6x_{n-1} = 5(2^{n+1} + 3^{n-1}) - 6(2^{n-1+1} + 3^{n-1-1}) \\
&= 2^n(10-6) + 3^{n-2}(15-6) = 2^{n+2} + 3^n,
\end{align*}
so that \(p(n+1)\) is true. It now follows from the Principle of induction (Strong form) that (5) holds for every \(n \in \mathbb{N}\).

Example 1.2.6. Suppose that \(n \in \mathbb{N}\) and \(n > 1\). Then \(n\) is representable as a product of primes. To prove this, let \(p(n)\) denote the statement
\[(n = 1) \text{ or } (n \text{ is a product of primes}).\]
First of all, clearly \(p(1)\) is true. Also 2 is a prime, and so is a product of primes, so that \(p(2)\) is true. Suppose now that \(n > 2\) and that \(p(m)\) is true for every \(1 \leq m < n\). Then in particular, every \(m \in \mathbb{N}\) satisfying \(2 \leq m < n\) is representable as a product of primes. If \(n\) is a prime, then it is obviously representable as a product of primes. If \(n\) is not a prime, then there exist \(n_1, n_2 \in \mathbb{N}\) satisfying \(2 \leq n_1 < n\) and \(2 \leq n_2 < n\) such that \(n = n_1n_2\). By our induction hypothesis, both \(n_1\) and \(n_2\) are representable as products of primes, so that \(n\) must be representable as a product of primes, whence \(p(n)\) is true. It now follows from the Principle of induction (Strong form) that every natural number \(n > 1\) is representable as a product of primes.

1.3. Completeness of the Real Numbers

The set \(\mathbb{Z}\) of all integers is an extension of the set \(\mathbb{N}\) of all natural numbers to include 0 and all numbers of the form \(-n\), where \(n \in \mathbb{N}\). The set \(\mathbb{Q}\) of all rational numbers is the set of all real numbers of the form \(pq^{-1}\), where \(p \in \mathbb{Z}\) and \(q \in \mathbb{N}\).
We see that the Field axioms and Order axioms hold good if the set $\mathbb{R}$ is replaced by the set $\mathbb{Q}$. On the other hand, the set $\mathbb{Q}$ is incomplete. A good illustration is the following well known result.

**Proposition 1A.** No rational number $x \in \mathbb{Q}$ satisfies $x^2 = 2$.

**Proof.** Suppose that $pq^{-1}$ has square 2, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may assume, without loss of generality, that $p$ and $q$ have no common factors apart from ±1. Then $p^2 = 2q^2$ is even, so that $p$ is even. We can write $p = 2r$, where $r \in \mathbb{Z}$. Then $q^2 = 2r^2$ is even, so that $q$ is even, contradicting that assumption that $p$ and $q$ have no common factors apart from ±1.

It follows that the real number we know as $\sqrt{2}$ does not belong to $\mathbb{Q}$. We shall now discuss a property that distinguishes the set $\mathbb{R}$ from the set $\mathbb{Q}$.

**Definition.** A non-empty set $S$ of real numbers is said to be bounded above if there exists a number $K \in \mathbb{R}$ such that $x \leq K$ for every $x \in S$. The number $K$ is called an upper bound of the set $S$.

**Completeness Axiom.** Suppose that $S$ is a non-empty set of real numbers and $S$ is bounded above. Then there is a real number $M \in \mathbb{R}$ such that $M \leq K$ for every upper bound $K$ of the set $S$, and that $M > L$ for any real number $L$ that is not an upper bound of $S$.

**Remark.** The crucial assertion is that this number $M$ is a real number. The set $S = \{ x \in \mathbb{Q} : x^2 < 2 \}$ is bounded above. We can take $K = 2$ or $K = 5^{2000}$. However, we clearly have $M = \sqrt{2}$.

### 1.4. Further Discussion on the Real Numbers

In this optional section, we shall first of all demonstrate the equivalence of the condition (WO) and the two forms of the Principle of induction.

**Proof of the equivalence of the Well-ordering principle and the two Principles of induction.** Our first step is to show that the condition (WO) is equivalent to the Principle of induction (strong form) (PIS).

\[ ((WO) \Rightarrow (PIS)) \]

Suppose that the conclusion of (PIS) does not hold. Then the subset

\[ S = \{ n \in \mathbb{N} : p(n) \text{ is false} \} \]

of $\mathbb{N}$ is non-empty. By (WO), $S$ has a least element, $n_0$ say. If $n_0 = 1$, then clearly (PIS1) does not hold. If $n_0 > 1$, then $p(m)$ is true for all $m \leq n_0 - 1$ but $p(n_0)$ is false, contradicting (PIS2).

\[ ((PIS) \Rightarrow (WO)) \]

Suppose that a non-empty subset $S$ of $\mathbb{N}$ does not have a least element. Consider the statement $p(n)$, given by $n \notin S$. Then $p(1)$ is true, otherwise 1 would be the least element of $S$. Suppose next that $p(m)$ is true for every natural number $m \leq n$, so that none of the numbers 1, 2, 3, ..., $n$ belongs to $S$. Then $p(n + 1)$ must also be true, for otherwise $n + 1$ would be the least element of $S$. It now follows from (PIS) that $S$ does not contain any element of $\mathbb{N}$, contradicting the assumption that $S$ is a non-empty subset of $\mathbb{N}$.

Next, we complete the proof by showing that the Principle of induction (weak form) (PIW) is equivalent to the Principle of induction (strong form) (PIS).

\[ ((PIS) \Rightarrow (PIW)) \]

Suppose that (PIW1) and (PIW2) both hold. Then clearly (PIS1) holds, since it is the same as (PIW1). On the other hand, if $p(m)$ is true for all $m \leq n$, then $p(n)$ is true in particular, so it follows from (PIW2) that $p(n + 1)$ is true, and this gives (PIS2). It now follows from (PIS) that $p(n)$ is true for every $n \in \mathbb{N}$.
((PIW) ⇒ (PIS)) Suppose that (PIS1) and (PIS2) both hold for a statement \( p(.) \). Consider a statement \( q(.) \), where \( q(n) \) denotes the statement \( p(m) \) is true for every \( m \leq n \).

Then the two conditions (PIS1) and (PIS2) for the statement \( p(.) \) imply respectively the two conditions (PIW1) and (PIW2) for the statement \( q(.) \). It follows from (PIW) that \( q(n) \) is true for every \( n \in \mathbb{N} \), and this clearly implies that \( p(n) \) is true for every \( n \in \mathbb{N} \).

We next discuss the completeness of the real numbers in greater detail. First of all, the Completeness axiom can be stated in the following alternative way.

**Completeness Axiom.** Suppose that \( S \) is a non-empty set of real numbers and \( S \) is bounded above. Then there is a real number \( M \in \mathbb{R} \) satisfying the following two conditions:

- (S1) For every \( x \in S \), the inequality \( x \leq M \) holds.
- (S2) For every \( \epsilon > 0 \), there exists \( x \in S \) such that \( x > M - \epsilon \).

**Remark.** It is not difficult to prove that the number \( M \) above is unique. It is also easy to deduce that if \( S \) is a non-empty set of real numbers and \( S \) is bounded below, then there is a unique real number \( m \in \mathbb{R} \) satisfying the following two conditions:

- (I1) For every \( x \in S \), the inequality \( x \geq m \) holds.
- (I2) For every \( \epsilon > 0 \), there exists \( x \in S \) such that \( x < m + \epsilon \).

**Definition.** The real number \( M \) satisfying conditions (S1) and (S2) is called the supremum of the non-empty set \( S \), and denoted by \( M = \sup S \). The real number \( m \) satisfying conditions (I1) and (I2) is called the infimum of the non-empty set \( S \), and denoted by \( m = \inf S \).

Let us now try to understand how numbers like \( \sqrt{2} \) fits into this setting. Recall that there is no rational number which satisfies the equation \( x^2 = 2 \). This means that the number that we know as \( \sqrt{2} \) is not a rational number. We now want to show that it is a real number. Let

\[ S = \{ x \in \mathbb{R} : x^2 < 2 \} . \]

Clearly the set \( S \) is non-empty, since \( 0 \in S \). On the other hand, the set \( S \) is bounded above; for example, it is not difficult to show that if \( x \in S \), then we must have \( x \leq 2 \); for if \( x > 2 \), then we must have \( x^2 > 4 \), so that \( x \not\in S \). Hence \( S \) is a non-empty set of real numbers and \( S \) is bounded above. It follows from the Completeness axiom that there is a real number \( M \) satisfying conditions (S1) and (S2). We now claim that \( M^2 = 2 \).

Suppose on the contrary that \( M^2 \neq 2 \). Then it follows from axiom (O1) that \( M^2 < 2 \) or \( M^2 > 2 \). Let us investigate these two cases separately.

If \( M^2 < 2 \), then we have

\[(M + \epsilon)^2 = M^2 + 2M \epsilon + \epsilon^2 < 2 \quad \text{whenever} \quad \epsilon < \min \left\{ 1, \frac{2 - M^2}{2M + 1} \right\} .\]

This means that \( M + \epsilon \in S \), contradicting condition (S1).

If \( M^2 > 2 \), then we have

\[(M - \epsilon)^2 = M^2 - 2M \epsilon + \epsilon^2 > 2 \quad \text{whenever} \quad \epsilon < \frac{M^2 - 2}{2M} .\]

This implies that any \( x > M - \epsilon \) will not belong to \( S \), contradicting condition (S2).

Note that \( M^2 = 2 \) and \( M \) is a real number. It follows that what we know as \( \sqrt{2} \) is a real number.
1.5. The Complex Numbers

It is easy to see that the equation $x^2 + 1 = 0$ has no solution $x \in \mathbb{R}$. In order to “solve” this equation, we have to introduce extra numbers into our number system.

Define the number $i$ by $i^2 + 1 = 0$. We then extend the field of all real numbers by adjoining the number $i$, which is then combined with the real numbers by the operations addition and multiplication in accordance with the Field axioms in Section 1.1. The numbers $a + bi$, where $a, b \in \mathbb{R}$, of the extended field are then added and multiplied in accordance with the Field axioms, suitably extended, and the restriction $i^2 + 1 = 0$. Note that the number $a + 0i$, where $a \in \mathbb{R}$, behaves like the real number $a$.

Remark. What we have said in the last paragraph basically amounts to the following. Consider two complex numbers $a + bi$ and $c + di$, where $a, b, c, d \in \mathbb{R}$. We have the addition rule

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and the multiplication rule

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

A simple consequence is the subtraction rule

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

For the division rule, suppose that $c + di \neq 0$, so that $c \neq 0$ or $d \neq 0$, whence $c^2 + d^2 \neq 0$. If

$$\frac{a + bi}{c + di} = x + yi,$$

where $x, y \in \mathbb{R}$, then

$$a + bi = (c + di)(x + yi) = (cx - dy) + (cy + dx)i.$$

It follows that

$$a = cx - dy,$$
$$b = cy + dx.$$

This system of simultaneous linear equations has the unique solution

$$x = \frac{ac + bd}{c^2 + d^2} \quad \text{and} \quad y = \frac{bc - ad}{c^2 + d^2},$$

so that

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.$$

The special case $a = 1$ and $b = 0$ gives

$$\frac{1}{c + di} = \frac{c - di}{c^2 + d^2}.$$

This can also be obtained by noting that $(c + di)(c - di) = c^2 + d^2$, so that

$$\frac{1}{c + di} = \frac{c - di}{(c + di)(c - di)} = \frac{c - di}{c^2 + d^2}.$$

It is also useful to note that $i^n$ has exactly four possible values, with $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. 

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Definition. Suppose that \( z = x + yi \), where \( x, y \in \mathbb{R} \). The real number \( x \) is called the real part of \( z \), and denoted by \( x = \Re z \). The real number \( y \) is called the imaginary part of \( z \), and denoted by \( y = \Im z \). The set \( \mathbb{C} = \{ z = x + yi : x, y \in \mathbb{R} \} \) is called the set of all complex numbers.

Example 1.5.1. We have
\[
\frac{(1 + 2i)^2}{1 - i} = \frac{-3 + 4i}{1 - i} = \frac{(-3 + 4i)(1 + i)}{(1 - i)(1 + i)} = \frac{-7 + i}{2} = \frac{-7}{2} + \frac{1}{2}i.
\]
Hence
\[
\Re \left( \frac{(1 + 2i)^2}{1 - i} \right) = -\frac{7}{2} \quad \text{and} \quad \Im \left( \frac{(1 + 2i)^2}{1 - i} \right) = \frac{1}{2}.
\]

Example 1.5.2. We have \( 1 + i + i^2 + i^3 = 0 \) and \( 5 + 7i^{2003} = 5 - 7i \).

Definition. Suppose that \( z = x + yi \), where \( x, y \in \mathbb{R} \). Then the complex number \( \overline{z} = x - yi \) is called the conjugate of the complex number \( z \).

**Proposition 1B.** Suppose that \( z \in \mathbb{C} \). Then
\[
\Re z = \frac{z + \overline{z}}{2} \quad \text{and} \quad \Im z = \frac{z - \overline{z}}{2i}.
\]

**Proof.** Write \( z = x + yi \), where \( x, y \in \mathbb{R} \). Then
\[
\frac{z + \overline{z}}{2} = \frac{(x + yi) + (x - yi)}{2} = x \quad \text{and} \quad \frac{z - \overline{z}}{2i} = \frac{(x + yi) - (x - yi)}{2i} = y
\]
as required. \( \Box \)

**Proposition 1C.** Suppose that \( z, w \in \mathbb{C} \). Then
\[
\overline{z + w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{zw} = \overline{z} \overline{w}.
\]

**Proof.** Write \( z = x + yi \) and \( w = u + vi \), where \( x, y, u, v \in \mathbb{R} \). Then
\[
\overline{z + w} = \overline{(x + u) + (y + v)i} = (x + u) - (y + v)i = (x - yi) + (u - vi) = \overline{z} + \overline{w}
\]
and
\[
\overline{zw} = \overline{(x + yi)(u + vi)} = \overline{(xu - yv) + (xv + yu)i} = (xu - yv) - (xv + yu)i = (x - yi)(u - vi) = \overline{z} \overline{w}
\]
as required. \( \Box \)


1.6. Polar Coordinates

Since every complex number is of the form \( z = x + yi \), where \( x, y \in \mathbb{R} \), we can identify \( z \) with the point \((x, y)\) on the \( xy\)-plane \( \mathbb{R}^2 \) as shown in the Argand diagram below:

![Argand diagram](image)

Note that the numbers \( z = x + yi \) and \( w = u + vi \), where \( x, y, u, v \in \mathbb{R} \), are represented by the points \((x, y)\) and \((u, v)\) respectively, and that their sum \( z + w \) is represented by the point \((x + u, y + v)\), the vertex opposite the vertex \((0, 0)\) in a parallelogram with \((x, y)\) and \((u, v)\) also as vertices. We sometimes say that addition of complex numbers satisfies the parallelogram law.

To describe a product in an Argand diagram is not as straightforward. Suppose that \( z = x + yi \), where \( x, y \in \mathbb{R} \). Consider the following Argand diagram:

![Argand diagram](image)

We shall study more carefully the triangle shown. By Pythagoras’s theorem, we have

\[ r^2 = x^2 + y^2. \]  

(6)

Also

\[ x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \]

(7)

**Definition.** Suppose that \( z = x + yi \), where \( x, y \in \mathbb{R} \). We write

\[ |z| = \sqrt{x^2 + y^2} \]

and call this the modulus of \( z \). On the other hand, any number \( \theta \in \mathbb{R} \) satisfying the equations (7) is called an argument of \( z \), and denoted by \( \arg z \).
Remarks. (1) Note that for a given \( z \in \mathbb{C} \), \( \text{arg} \ z \) is not unique. Clearly we can add any integer multiple of \( 2\pi \) to \( \theta \) without affecting (7). We sometimes call a real number \( \theta \in \mathbb{R} \) the principal argument of \( z \) if \( \theta \) satisfies the equations (7) and \( -\pi < \theta \leq \pi \). Note that it follows from (7) that \( y/x = \tan \theta \). However, even with this restriction on \( \theta \), it is not meaningful to write \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \).

(2) Suppose that \( y = 0 \), so that \( z = x \in \mathbb{R} \). Then

\[
|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}
\]

and this is simply the absolute value of the real number \( x \).

(3) In view of Remark (1) above, we need to exercise extreme care when we try to determine an angle \( \theta \) which satisfies the equations (7). The best advice is always to place the complex number \( z \) on the Argand diagram and determine first of all a suitable range for \( \theta \). For example, we know that if \( z = -1 - i \), then a suitable range for \( \theta \) may be \( \pi < \theta < 3\pi/2 \) or \( -\pi < \theta < -\pi/2 \). Once such a suitable range is determined, the equation (8) will have a unique solution \( \theta \) within this range.

Definition. Suppose that \( z = x + yi \neq 0 \), where \( x, y \in \mathbb{R} \). Suppose further that the numbers \( r, \theta \in \mathbb{R} \) satisfy (6) and (7), and that \( r > 0 \) and \( -\pi < \theta \leq \pi \). Then we say that the pair \((r, \theta)\) form the polar coordinates of \( z \).

Remarks. (1) In view of (7), we have \( z = r(\cos \theta + i \sin \theta) \).

(2) Often, we write \( e^{i\theta} = \cos \theta + i \sin \theta \). However, this is presupposing that we have understood the exponential function with complex exponents.

Example 1.6.1. Suppose that \( z = 1 + i \). Then \(|z| = \sqrt{2}\) and \( \text{arg} \ z = \pi/4 \). Note also that

\[
z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).
\]

Try to draw the Argand diagram.

Example 1.6.2. The polar coordinates \((2, -2\pi/3)\) represent the complex number

\[
w = 2\cos \left( -\frac{2\pi}{3} \right) + 2i \sin \left( -\frac{2\pi}{3} \right) = -1 - i\sqrt{3}.
\]

Try to draw the Argand diagram.

The modulus has three very important properties that we often use.

**Proposition 1D.**

(a) For every \( z \in \mathbb{C} \), we have \(|z|^2 = z\bar{z}\).

(b) For every \( z, w \in \mathbb{C} \), we have \(|zw| = |z||w|\).

(c) For every \( z, w \in \mathbb{C} \), we have \(|z + w| \leq |z| + |w|\).
PROOF. (a) Write $z = x + yi$, where $x, y \in \mathbb{R}$. Then $z\overline{z} = (x + yi)(x - yi) = x^2 + y^2$.

(b) Write $z = x + yi$ and $w = u + vi$, where $x, y, u, v \in \mathbb{R}$. Then $zw = (xu - yv) + (xv + yu)i$, so that
\[|zw|^2 = (xu - yv)^2 + (xv + yu)^2 = (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2.\]
The result follows on taking square roots.

(c) Note that the result is trivial if $z + w = 0$. Suppose now that $z + w \neq 0$. Then
\[
\frac{|z| + |w|}{|z + w|} = \frac{|z|}{|z + w|} + \frac{|w|}{|z + w|} \geq \Re \frac{z}{z + w} + \Re \frac{w}{z + w} = \Re \left(\frac{z}{z + w} + \frac{w}{z + w}\right) = \Re 1 = 1.
\]
The result follows immediately.  

REMARK. Proposition 1D(c) is known as the Triangle inequality. It can be understood easily from the diagram below:

The inequality follows on noticing that the sum of the lengths of two sides of a triangle is at least the length of the third side.

We have shown earlier that the cartesian coordinates $(x, y)$ are very useful for adding two complex numbers, whereas multiplication of complex numbers has a rather messy formula in cartesian coordinates. Let us use polar coordinates instead.

Suppose that
\[z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad w = s(\cos \phi + i \sin \phi),\]
where $r, s, \theta, \phi \in \mathbb{R}$ and $r, s > 0$. Then
\[
zw = rs(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)
= rs((\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi))
= rs(\cos(\theta + \phi) + i \sin(\theta + \phi)).
\]
(9)

It follows that if we represent complex numbers in polar coordinates, then multiplication of complex numbers simply means essentially multiplying the moduli and adding the arguments. On the other hand, it is not difficult to show that
\[
\frac{z}{w} = \frac{r}{s}(\cos(\theta - \phi) + i \sin(\theta - \phi)).
\]
(10)
Example 1.6.3. Suppose that \( z = 1 + i \) and \( w = -1 - i \sqrt{3} \). Since
\[
z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \quad \text{and} \quad w = 2 \left( \cos \left( -\frac{2\pi}{3} \right) + i \sin \left( -\frac{2\pi}{3} \right) \right),
\]
it follows from (9) that
\[
zw = 2\sqrt{2} \left( \cos \left( \frac{\pi}{4} - \frac{2\pi}{3} \right) + i \sin \left( \frac{\pi}{4} - \frac{2\pi}{3} \right) \right) = 2\sqrt{2} \left( \cos \left( -\frac{5\pi}{12} \right) + i \sin \left( -\frac{5\pi}{12} \right) \right).
\]
Note also that
\[
zw = (1 + i)(-1 - i \sqrt{3}) = (\sqrt{3} - 1) - i(\sqrt{3} + 1),
\]
so that
\[
\cos \left( \frac{\pi}{4} - \frac{2\pi}{3} \right) = \frac{\sqrt{3} - 1}{2\sqrt{2}} \quad \text{and} \quad \sin \left( \frac{\pi}{4} - \frac{2\pi}{3} \right) = -\frac{\sqrt{3} + 1}{2\sqrt{2}}.
\]
On the other hand, it follows from (10) that
\[
\frac{z}{w} = \frac{\sqrt{2}}{2} \left( \cos \left( \frac{\pi}{4} + \frac{2\pi}{3} \right) + i \sin \left( \frac{\pi}{4} + \frac{2\pi}{3} \right) \right) = \frac{1}{\sqrt{2}} \left( \cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12} \right).
\]

Example 1.6.4. Suppose that \( z = 1 + i \). Then repeated application of (9) yields
\[
z^5 = 4\sqrt{2} \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = 4\sqrt{2} \left( \cos \left( -\frac{3\pi}{4} \right) + i \sin \left( -\frac{3\pi}{4} \right) \right).
\]
Note that we have to subtract \( 2\pi \) to get the principal argument of \( z^5 \).

Our last example suggests the following important result.

**Proposition 1E.** (De Moivre’s Theorem) Suppose that \( n \in \mathbb{N} \) and \( \theta \in \mathbb{R} \). Then
\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.
\]

**Proof.** This follows from repeated application of the product formula in polar coordinates to the complex number \( z = \cos \theta + i \sin \theta \), noting that \( |z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1 \).

**Remarks.**
1. Formally, Proposition 1E is proved by induction; see Example 1.2.4.
2. In the notation \( e^{i\theta} = \cos \theta + i \sin \theta \), de Moivre’s theorem is the observation that \( e^{in\theta} = (e^{i\theta})^n \).

Example 1.6.5. We have
\[
\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3 \cos^2 \theta \sin \theta + 3i \cos \theta \sin^2 \theta + i^3 \sin^3 \theta
\]
\[
= \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta).
\]
It follows that
\[
\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \quad \text{and} \quad \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.
\]

**Remark.** It can be shown that the conclusion of de Moivre’s theorem remains true for every \( n \in \mathbb{Q} \).
1.7. Finding Roots

Let us try to find the square roots of the complex number $a + bi$, where $a, b \in \mathbb{R}$. We are therefore looking for complex numbers $x + yi$, where $x, y \in \mathbb{R}$ and

$$(x + yi)^2 = a + bi.$$ 

We may assume that $b \neq 0$, otherwise the solution is trivial. Since $(x + yi)^2 = (x^2 - y^2) + 2xyi$, we must have

$$x^2 - y^2 = a, \quad (11)$$
$$2xy = b. \quad (12)$$

It follows from (11) and (12) that

$$x^2 + y^2 = \sqrt{a^2 + b^2},$$

where the square root is non-negative. Together with (11), we obtain

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2} \quad \text{and} \quad y^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}. \quad (13)$$

Note that the equations (13) generally yield two solutions for $x$ and two solutions for $y$. However, note that by (12), the product $xy$ has to have the same sign as $b$. It follows that

$$\sqrt{a + bi} = \pm \left( \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i \frac{b}{|b|} \sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}} \right),$$

where the square roots are non-negative.

This is a rather cumbersome approach, and is not to be recommended for higher order roots. As we have shown earlier, it is more convenient to do multiplication of complex numbers in polar coordinates, so let us attempt to find roots using polar coordinates.

Suppose that $c = R(\cos \alpha + i \sin \alpha)$, where $c, \alpha \in \mathbb{R}$ and $c > 0$. Consider the equation

$$z^n = c,$$

where $n \in \mathbb{N}$ is fixed. Writing $z = r(\cos \theta + i \sin \theta)$, where $r, \theta \in \mathbb{R}$ and $r > 0$, we have, using de Moivre’s theorem, that

$$z^n = r^n(\cos n\theta + i \sin n\theta) = R(\cos \alpha + i \sin \alpha).$$

It follows that

$$r^n = R,$$

and we can take

$$n\theta = \alpha + 2k\pi, \quad \text{where} \quad k = 0, 1, \ldots, n - 1,$$

so that

$$\theta = \frac{\alpha + 2k\pi}{n}, \quad \text{where} \quad k = 0, 1, \ldots, n - 1. \quad (14)$$
Note that no two values of $\theta$ in (14) differ by an integer multiple of $2\pi$. It follows that

$$z = \sqrt[n]{R} \left( \cos \frac{\alpha + 2k\pi}{n} + i \sin \frac{\alpha + 2k\pi}{n} \right), \quad \text{where } k = 0, 1, \ldots, n - 1,$$

(15)
give $n$ distinct complex numbers. On the other hand, it follows from (15) and de Moivre’s theorem that each of the $n$ numbers in (15) satisfies $z^n = c$.

We have proved the following result.

**PROPOSITION 1F.** Suppose that $c = R(\cos \alpha + i \sin \alpha)$, where $c, \alpha \in \mathbb{R}$ and $c > 0$. Then the solutions of the equation $z^n = c$ are given by (15).

**EXAMPLE 1.7.1.** The 7-th roots of $1 - i$ can be calculated as follows. Note here that $c = 1 - i = \sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)$ (observe that it is not necessary to use the principal argument). It follows from Proposition 1F that the 7-th roots of $1 - i$ are given by

$$z = \sqrt[7]{2} \left( \cos \left( \frac{\pi}{4} + \frac{2k\pi}{7} \right) + i \sin \left( \frac{\pi}{4} + \frac{2k\pi}{7} \right) \right), \quad \text{where } k = 0, 1, 2, 3, 4, 5, 6.$$

**EXAMPLE 1.7.2.** The case $c = 1$ is particularly important, as we get the $n$-th roots of 1. Note that $R = 1$ and $\alpha = 0$. It follows that the $n$-th roots of unity are given by

$$z = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad \text{where } k = 0, 1, \ldots, n - 1.$$

**EXAMPLE 1.7.3.** Consider the polynomial $p(z) = z^3 - z^2 + 2z - 2$, and observe that $z = 1$ is a root. Furthermore,

$$p(z) = (z - 1)(z^2 + 2),$$

so that two other solutions are given by the roots of the equation $z^2 = -2$. It is easy to see that $-2 = 2(\cos \pi + i \sin \pi)$ in polar form. It follows that the two roots of $z^2 = -2$ are given by

$$z = \sqrt{2} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = \sqrt{2}i \quad \text{and} \quad z = \sqrt{2} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) = -\sqrt{2}i.$$

**EXAMPLE 1.7.4.** Consider the polynomial $p(z) = z^6 - 2z^3 + 4 = 0$. Writing $w = z^3$, we then have $w^2 - 2w + 4 = 0$, with roots

$$w = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{-3} = 1 \pm \sqrt{3}i.$$

To find the roots of $p(z)$, we have to find all the roots of

$$z^3 = 1 + \sqrt{3}i,$$

as well as all the roots of

$$z^3 = 1 - \sqrt{3}i.$$
To study (16), note that $1 + \sqrt{3}i = 2(\cos(\pi/3) + i\sin(\pi/3))$. It follows from Proposition 1F that the roots of (16) are given by

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right)\right),$$

where $k = 0, 1, 2$; in other words,

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right), \quad z_2 = \sqrt{2}\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right), \quad z_3 = \sqrt{2}\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right).$$

To study (17), note that $1 - \sqrt{3}i = 2(\cos(5\pi/3) + i\sin(5\pi/3))$. It follows from Proposition 1F that the roots of (17) are given by

$$z = \sqrt{2}\left(\cos\left(\frac{5\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{5\pi}{9} + \frac{2k\pi}{3}\right)\right),$$

where $k = 0, 1, 2$; in other words,

$$z_4 = \sqrt{2}\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right), \quad z_5 = \sqrt{2}\left(\cos\frac{11\pi}{9} + i\sin\frac{11\pi}{9}\right), \quad z_6 = \sqrt{2}\left(\cos\frac{17\pi}{9} + i\sin\frac{17\pi}{9}\right).$$

### 1.8. Analytic Geometry

In classical analytic geometry, we express the equation of a locus as a relation between $x$ and $y$. If we write $z = x + iy$, then such an equation can be equally well described as a relation between $z$ and $\pi$. However, it is important to bear in mind that a complex equation is usually equivalent to two real equations, since each of the real part and the imaginary part of the complex equation gives rise to a real equation. It follows that to obtain a genuine locus, these two equations should be essentially the same. We also study some simple regions on the complex plane.

Here, we shall restrict our discussion to three examples. The reader is advised to draw some pictures.

**Example 1.8.1.** The equation of a circle can be given by

$$|z - c| = r. \quad (18)$$

To see this, suppose that $z = x + iy$ and $c = a + ib$, where $x, y, a, b \in \mathbb{R}$. Then

$$|z - c|^2 = |(x + iy) - (a + ib)|^2 = |(x - a) + i(y - b)|^2 = (x - a)^2 + (y - b)^2,$$

so that we have the equation $(x - a)^2 + (y - b)^2 = r^2$. Note that the equation (18) can also be written in the form

$$(z - c)(\bar{z} - \bar{c}) = r^2. \quad (19)$$

Note also that equation (19) is in invariant under conjugation; in other words, the conjugate of (19) is exactly the same as (19). Next, we consider the inequality $|z - c| < r$. A similar argument as above leads to the inequality $(x - a)^2 + (y - b)^2 < r^2$. This represents the region on the $xy$-plane inside the circle $(x - a)^2 + (y - b)^2 = r^2$. Similarly, the inequality $|z - c| > r$ represents the region on the $xy$-plane outside the circle $(x - a)^2 + (y - b)^2 = r^2$. 

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*Chapter 1 : The Number System*
Example 1.8.2. The equation

\[ |z - 1| = |z + 1| \]  \hspace{1cm} (20)

represents a straight line. For writing \( z = x + iy \), where \( x, y \in \mathbb{R} \), equation (20) becomes

\[ |(x - 1) + iy| = |(x + 1) + iy|, \]

so that squaring both sides, we obtain

\[ (x - 1)^2 + y^2 = (x + 1)^2 + y^2. \]

On simplifying, we obtain \( x = 0 \). Interpreted geometrically, note that \( |z - 1| \) represents the distance between the points \( z \) and 1 on the Argand plane, while \( |z + 1| \) represents the distance between the points \( z \) and \(-1\) on the Argand plane. Equation (20) thus asserts that \( z \) is equidistant from 1 and from \(-1\). To achieve this, \( z \) must lie on the \( y \)-axis; in other words, we must have \( x = 0 \). Next, we consider the inequality \( |z - 1| < |z + 1| \). This is the region on the complex plane containing all points \( z \) such that the distance of \( z \) from 1 is smaller than the distance of \( z \) from \(-1\). A little thought leads to the half plane \( x > 0 \); in other words, the right half of the complex plane.

Example 1.8.3. Consider a parallelogram \( OABC \), where \( \overline{OB} \) is a diagonal and \( \overline{AC} \) is the other diagonal. We now place the parallelogram on the Argand plane so that the vertex \( O \) is precisely at the point 0. Suppose that the points \( A \) and \( C \) are represented by the complex numbers \( z \) and \( w \) respectively. Then the vertex \( B \) is represented by the complex number \( z + w \). It is not difficult to see that the midpoint of the diagonal \( \overline{OB} \) is represented by the complex number

\[ \frac{1}{2} (0 + (z + w)) = \frac{z + w}{2}, \]

which also represents the midpoint of the diagonal \( \overline{AC} \). This proves that the two diagonals of a parallelogram bisect each other.
Problems for Chapter 1

1. Suppose that \(a, b, c, d\) are positive real numbers satisfying \(a < b\) and \(c < d\). Show that \(ac < bd\).
   [Hint: Use the Field axioms and the Order axioms only.]

2. Find \(x, y \in \mathbb{R}\) such that \(x < y\) and \(x^{-1} < y^{-1}\).

3. Suppose that \(x, y, z \in \mathbb{R}\). Use the Field axioms and the Order axioms only to show that
   a) if \(x + z = y + z\), then \(x = y\);
   b) if \(z \neq 0\) and \(xz = yz\), then \(x = y\);
   c) if \(xy = 0\), then \(x = 0\) or \(y = 0\).

4. Show that if \(x, y, a \in \mathbb{R}\) satisfy \(x < y\) and \(a < 0\), then \(ax > ay\).
   [Hint: Use the Field axioms and the Order axioms only.]

5. Prove that \(\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \ldots + \frac{n}{3} = \frac{1}{4}n^2(n + 1)^2\) for every \(n \in \mathbb{N}\).

6. Prove that \(2^n > n^3\) for every natural number \(n > 9\).

7. Prove that for every \(n \in \mathbb{N}\), \(5^{2n} - 6n + 8\) is divisible by 9.

8. Complex numbers are numbers of the form \(a + bi\), where \(a, b \in \mathbb{R}\) and \(i^2 = -1\). Such numbers can be represented on the \(xy\)-plane by points of the form \((a, b)\).
   a) Consider the complex number \(3 + 4i\), represented on the \(xy\)-plane by the point \((3, 4)\). Draw a picture of the \(xy\)-plane, clearly indicating the origin \((0, 0)\) and the point \((3, 4)\).
   b) Draw a line segment joining \((0, 0)\) and \((3, 4)\). What is the length of this line segment?
   c) We now multiply the complex number \(3 + 4i\) by the complex number \(i\) to obtain the product \(i(3 + 4i)\). Which point on the \(xy\)-plane represents the number \(i(3 + 4i)\)? Indicate this point on the \(xy\)-plane.
   d) Draw a line segment joining \((0, 0)\) and the point in part (c). What is the length of this line segment? What is the angle between this line segment and the line segment in part (b)?
   e) Which point on the \(xy\)-plane represents the number \(2i(3+4i)\)? Indicate this point on the \(xy\)-plane. Draw a line segment joining \((0, 0)\) and this point. What is the length of this line segment? What is the angle between this line segment and the line segment in part (b)?
   f) Let \(a, b \in \mathbb{R}\) be fixed positive real numbers. Repeat steps (a)–(e) with \(a + bi\) in place of \(3 + 4i\).
   g) What can we say about the effect of multiplying a complex number by \(i\)?
   h) What can we say about the effect of multiplying a complex number by \(2i\)?

9. Let \(z_1 = 2 + 4i\) and \(z_2 = \frac{1}{2}(1 - 5i)\). Find each of the following numbers:
   a) \(z_1z_2\)
   b) \((z_1 + 2z_2)^2\)
   c) \(iz_1 + 2z_2\)

10. Let \(z_1 = 5\), \(z_2 = 3 + 4i\) and \(z_3 = 1 - \sqrt{3}i\). Evaluate each of the following numbers:
    a) \(\overline{z_1}\)
    b) \(\overline{z_2}\)
    c) \(z_1 - z_2\)
    d) \(z_2z_3\)
    e) \(\Re(z_1/z_2)\)
    f) \(\arg z_3\)

11. Suppose that \(z = x + iy\), where \(x, y \in \mathbb{R}\) and \(i^2 = -1\). Write down each of the following numbers in terms of \(x\) and \(y\):
    a) \(\Re(z)\)
    b) \(\Im(z)\)
    c) \(\overline{z}\)
    d) \(|z|^2\)
    e) \(\Im(z^{-1})\)
    f) \(\frac{z\overline{z}}{|z|^2}\)

12. Express each of the following numbers in the form \(x + yi\), where \(x, y \in \mathbb{R}\):
    a) \((1 + 3i)^3\)
    b) \((3 - 2i)^2 - (3 + 2i)^2\)
    c) \((1 + i + i^2 + i^3)^{100}\)
13. Find the real and imaginary parts of each of the following numbers:
   a) \( \frac{1 + i}{1 - i} \),  
   b) \( \frac{1 + 2i}{3 + 4i} \),  
   c) \( \frac{x + yi}{1} \).

14. Suppose that \( a + bi = c + di \), where \( a, b, c, d \in \mathbb{R} \). Show that \( a = c \) and \( b = d \).

15. For each of the following complex numbers \( z \), find real numbers \( x \) and \( y \) such that \( z = x + iy \), then show the positions of \( z \) and \( \overline{z} \) on the Argand diagram, and determine the modulus and the principal argument of \( z \):
   a) \( z = (1 + i)^4 \)  
   b) \( z = \frac{3 + 4i}{1 - 2i} \).

16. Let \( z = x + iy \), where \( x, y \in \mathbb{R} \).
   a) Write down \( |z|^2 \) and \( (\Re z)^2 \).
   b) Hence prove that \( |z| \geq \Re z \).
   c) For what values of \( z \) does equality hold?

17. Let \( z = 3 - 4i \).
   a) Find \( \overline{z}, |z| \) and \( z^{-1} \).
   b) Verify that \( z^{-1} = \overline{z}/|z|^2 \).

18. Solve each of the following equations and leave your answers in rectangular form:
   a) \( z^2 + 4z + 5 = 0 \)  
   b) \( z^2 + iz - 1 = 0 \).

19. Solve the equation \( z^2 + z + 1 = 0 \). If the solutions are \( z_1 \) and \( z_2 \), calculate \( z_1^3 \) and \( z_2^3 \).

20. Consider the equation \( z^3 - 3z^2 + 4z - 2 = 0 \).
   a) Solve the equation.
   b) Are your solutions conjugates of each other?
   c) Comment on the results.

21. Find the square roots of \( 5 + 12i \) by taking the following steps:
   a) Rewrite the equation \( z^2 = 5 + 12i \) in real variables \( x \) and \( y \), where \( z = x + iy \).
   b) By considering the real and imaginary parts of your result in (a), solve for \( x \) and \( y \).

22. Consider the equation \( z^3 - 3z^2 + 4z - 2 = 0 \).
   a) Verify that \( 1 + i \) is a solution of the equation.
   b) Find also the other solutions.

23. You are given that \( z = 1 \) is a solution of the cubic equation \( z^3 - 5z^2 + 9z - 5 = 0 \). Find the other two solutions.

24. You are given that \( z = 2 \) is a solution of the cubic equation \( z^3 - 6z^2 + 13z - 10 = 0 \). Find the other two solutions.

25. You are given that \( z = -1 \) is a solution of the equation \( z^3 + 3z^2 + 6z + 4 = 0 \). Use this to find the other two solutions. Then indicate the positions of the three solutions in the Argand diagram.

26. Suppose that a non-zero complex number \( z \) has modulus \( r \) and argument \( \theta \). Write down the modulus and argument of each of the following:
   a) \( z \)  
   b) \( z^3 \)  
   c) \( z^{-1} \)  
   d) \( -z \)  
   e) \( z\overline{z} \)

27. Express each of the following in polar form:
   a) \( -7 + 7i \)  
   b) \( \sqrt{3} + 3i \)  
   c) \( -i \)  
   d) \( 1 + \sqrt{3}i \)  
   e) \( 1 - \sqrt{3}i \)  
   f) \( -2 - 2i \)
28. Express each of the following in cartesian form:
   a) \(2e^{\pi i/4}\)  
   b) \(e^{-\pi i}\)  
   c) \(3e^{2\pi i/3}\)  
   d) \(7e^{\pi i/6}\)  
   e) \(8e^{2\pi i}\)  
   f) \(9e^{-\pi i/4}\)

29. a) On the Argand diagram, choose a point \(z\) with positive real and imaginary parts and satisfying \(|z| = 2\). Then indicate the positions of \(\overline{z}\) and \(z^{-1}\).
   b) Explain in simple English how you come to your conclusions.
   c) What is the distance between \(z^{-1}\) and the origin 0?

30. Suppose that the complex number \(z\) satisfies \(|z| = 1\). Prove that \(z = z^{-1}\).

31. Let \(z\) be a non-zero complex number. Explain why 0, \(z^{-1}\) and \(\overline{z}\) lie in a straight line on the Argand plane.

32. Suppose that the complex number \(z_1\) is a cube root of unity and the complex number \(z_2\) is a 4-th root of unity. Let \(z = z_1z_2\). Show that \(z\) is a 12-th of unity.

33. Use de Moivre’s theorem to show that for every real number \(\theta\), we have \(\cos 2\theta = \cos^2 \theta - \sin^2 \theta\) and \(\sin 2\theta = 2\sin \theta \cos \theta\).

34. Consider the equation \(z^6 = -64\).
   a) Find the six roots of the equation and express them in polar form.
   b) Convert your answers in part (a) to rectangular form. Do not use your calculators to find the cosine and sine of the argument. Use instead the well known fact that \(\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}\) and \(\sin \frac{\pi}{6} = \frac{1}{2}\).
   c) Indicate the positions of the six roots in the Argand diagram.

35. For each of the following equations, find all the solutions:
   a) \(z^7 = 32\)  
   b) \(z^2 = 15 + 8i\)  
   c) \(\overline{z} = z^2\)  
   d) \(3\overline{z} = z^2\)

36. Let \(z = -1 - i\).
   a) Draw an Argand diagram clearly indicating the positions of \(z\) and \(\overline{z}\).
   b) Find the modulus \(|z|\).
   c) Find a non-negative real number \(r\) and an angle \(\theta\) satisfying \(0 < \theta < 2\pi\) and \(z = r(\cos \theta + i \sin \theta)\).
   d) Express \(z^3\) in polar form and then indicate its position in the Argand diagram you have drawn in part (a).

37. Consider the equation \(z^4 = -16\).
   a) By first writing \(z\) and \(-16\) in polar form, find all the four roots of the equation, expressing them in polar coordinates.
   b) Draw an Argand diagram clearly indicating the positions of the four roots. You do not need to calculate the rectangular coordinates of the roots.

38. Consider the equation \(z^6 - z^4 + 4z^2 - 4 = 0\).
   a) Show that \(z = \pm 1\) are solutions of the equation.
   b) Find the other four solutions of the equation, and express them in rectangular form.
   c) Draw an Argand diagram clearly showing all six solutions of the equation.

39. Find in polar form the cube roots of \(-2 - 2i\). Hence find a pair of rational numbers \(a\) and \(b\) such that \((a + bi)^3 = -2 - 2i\).

40. By writing \(z = x + iy\), where \(x, y \in \mathbb{R}\) and \(i^2 = -1\), show that the equation \(|z + 1| = |z - i|\) represents a straight line. What is the equation of this line?
41. Sketch the graph of \(|z + 3| - |z - 3| = 2\). What is it?

42. Prove that if \(z = x + yi\) and \(x > 10y > 0\) then \(z^{15}\) is in the first quadrant of the Argand diagram.

**Harder Problems for Chapter 1**

43. Let \(A = \{2^{-m} + 3^{-n} : m, n \in \mathbb{N}\}\). Find \(\sup A\) and \(\inf A\).

44. For each of the following sets, determine whether the set is bounded above, bounded below, both or neither. Find also the supremum and/or infimum where appropriate:
   a) \(\{x \in \mathbb{R} : x^3 - 4x < 0\}\)
   b) \(\{y : y = 2^{-x} \text{ where } x \in \mathbb{N}\}\)
   c) \(\{y : y = 1 + x^2 \text{ where } x \in \mathbb{R}\}\)

45. Suppose that \(a, b \in \mathbb{R}\) and \(a < b + n^{-1}\) for every \(n \in \mathbb{N}\). Show that \(a \leq b\).
   [Hint: Suppose on the contrary that \(a > b\). Try to obtain a contradiction.]

46. Suppose that \(A\) and \(B\) are two non-empty bounded sets of real numbers.
   a) Show that \(\sup(A \cup B) = \max\{\sup A, \sup B\}\).
   b) How about \(A \cap B\)?

47. a) Suppose that \(x \leq a\) for every \(x \in E\). Show that \(\sup E \leq a\).
   b) Show that the corresponding statement with \(\leq\) replaced by \(<\) does not hold.

48. Suppose that \(A\) and \(B\) are two non-empty bounded sets of real numbers. Suppose further that \(E = \{a + b : a \in A\ \text{and} \ b \in B\}\) and \(F = \{a - b : a \in A \ \text{and} \ b \in B\}\). Show that \(\sup E = \sup A + \sup B\) and \(\sup F = \sup A - \inf B\).

49. a) Suppose that \(A\) is a non-empty bounded set of real numbers. Suppose further that \(B\) is a non-empty subset of \(A\). Show that \(\inf A \leq \inf B \leq \sup B \leq \sup A\).
   b) Suppose that \(A\) is a non-empty set of real numbers bounded above, and that the real number \(b \geq 0\). Show that if \(C = \{bx : x \in A\}\), then \(\sup C = b \sup A\).
   c) Suppose that \(A\) and \(B\) are non-empty sets of positive real numbers bounded above. Show that if \(C = \{xy : x \in A \ \text{and} \ y \in B\}\), then \(\sup C = (\sup A)(\sup B)\).