7.1. Use of Cauchy’s Residue Theorem

In this section, we briefly discuss the use of contour integration to calculate Fourier transforms. Recall Example 6.1.3, where Cauchy’s integral theorem plays a role in the determination of the Fourier transform. Here we proceed along similar lines. However, the functions involved have singularities, so that Cauchy’s residue theorem is used instead. We illustrate our technique by looking at an example.

Example 7.1.1. Consider the function \( f : \mathbb{R} \to \mathbb{C} \), given for every \( x \in \mathbb{R} \) by
\[
f(x) = \frac{1}{x^4 + 1}.
\]
It is easy to see that \( f \in G(\mathbb{R}) \), with Fourier transform
\[
F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} e^{-iwx} \, dx.
\]
Note that
\[
\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} e^{-iwx} \, dx = \lim_{R \to \infty} \int_{L(R)} \frac{e^{-iwx}}{z^4 + 1} \, dz,
\]
where the straight line segment \( L(R) = [-R, R] \) on the complex plane \( \mathbb{C} \). The function
\[
g(z) = \frac{e^{-iwz}}{z^4 + 1}
\]
is analytic on the complex plane \( \mathbb{C} \), apart from simple poles at
\[
z = \pm 1 \pm i \sqrt{2}.
\]
Suppose first of all that \( w \geq 0 \). Let \( C^- (R) \) denote the bottom half of the circle of radius \( R \) and centred at the origin, followed in the anticlockwise direction. Cauchy’s residue theorem then gives, for large \( R \),
\[
\frac{1}{2\pi i} \int_{C^- (R)} \frac{e^{-iwz}}{z^4 + 1} \, dz - \frac{1}{2\pi i} \int_{L(R)} \frac{e^{-iwz}}{z^4 + 1} \, dz = \text{res} \left( g, \frac{1 - i}{\sqrt{2}} \right) + \text{res} \left( g, \frac{-1 - i}{\sqrt{2}} \right).
\]
Note that if \( z = x + iy \in C^- (R) \), where \( x, y \in \mathbb{R} \), then \( y < 0 \) and \( |e^{-iwz}| = e^{uw} \leq 1 \). It can then be checked easily that the integral over \( C^- (R) \) vanishes as \( R \to \infty \). On the other hand, standard calculation gives
\[
\text{res} \left( g, \frac{1 - i}{\sqrt{2}} \right) + \text{res} \left( g, \frac{-1 - i}{\sqrt{2}} \right) = \frac{i e^{-w/\sqrt{2}}}{2\sqrt{2}} \left( \cos \frac{w}{\sqrt{2}} + \sin \frac{w}{\sqrt{2}} \right).
\]
It follows that
\[
F(w) = \frac{e^{-w/\sqrt{2}}}{2\sqrt{2}} \left( \cos \frac{w}{\sqrt{2}} + \sin \frac{w}{\sqrt{2}} \right).
\]
Suppose next that \( w \leq 0 \). Let \( C^+(R) \) denote the top half of the circle of radius \( R \) and centred at the origin, followed in the anticlockwise direction. Cauchy’s residue theorem then gives, for large \( R \),

\[
\frac{1}{2\pi i} \int_{C^+(R)} \frac{e^{-iwz}}{z^4 + 1} \, dz + \frac{1}{2\pi i} \int_{L(R)} \frac{e^{-iwz}}{z^4 + 1} \, dz = \text{res} \left( g, \frac{1 + i}{\sqrt{2}} \right) + \text{res} \left( g, \frac{-1 + i}{\sqrt{2}} \right).
\]

Note that if \( z = x + iy \in C^+(R) \), where \( x, y \in \mathbb{R} \), then \( y > 0 \) and \( |e^{-iwz}| = e^{wy} \leq 1 \). It can then be checked easily that the integral over \( C^+(R) \) vanishes as \( R \to \infty \). On the other hand, standard calculation gives

\[
\text{res} \left( g, \frac{1 + i}{\sqrt{2}} \right) + \text{res} \left( g, \frac{-1 + i}{\sqrt{2}} \right) = \frac{ie^{-w/\sqrt{2}}}{2\sqrt{2}} \left( \sin \frac{w}{\sqrt{2}} - \cos \frac{w}{\sqrt{2}} \right).
\]

It follows that

\[
F(w) = \frac{e^{-w/\sqrt{2}}}{2\sqrt{2}} \left( \cos \frac{w}{\sqrt{2}} - \sin \frac{w}{\sqrt{2}} \right).
\]

We can summarize the ideas and establish the following result.

**Theorem 7.1.** Suppose that \( f \in G(\mathbb{R}) \), and that the domain of \( f \) can be extended to the whole complex plane \( \mathbb{C} \), resulting in a complex valued function analytic on \( \mathbb{C} \), apart from poles \( z_1', \ldots, z_m' \) on the lower half plane and \( z_1'', \ldots, z_n'' \) on the upper half plane. Suppose further that

\[
(7.1) \quad \lim_{R \to \infty} \left( R \max_{|z|=R} |f(z)| \right) = 0.
\]

Then the Fourier transform \( F \) of \( f \) is given by

\[
F(w) = \begin{cases} 
-i \sum_{j=1}^{m} \text{res}(f(z)e^{-iwz}, z_j'), & \text{if } w \geq 0, \\
+i \sum_{j=1}^{n} \text{res}(f(z)e^{-iwz}, z_j''), & \text{if } w \leq 0.
\end{cases}
\]

**Proof.** Note first of all that

\[
F(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-iwx} \, dx = \lim_{R \to \infty} \frac{1}{2\pi} \int_{L(R)} f(z)e^{-iwz} \, dz,
\]

where the straight line segment \( L(R) = [-R, R] \) on the complex plane \( \mathbb{C} \). Since the function \( e^{-iwz} \) is entire and non-zero on \( \mathbb{C} \), it follows that the function \( f(z)e^{-iwz} \) is analytic on \( \mathbb{C} \), apart from poles \( z_1', \ldots, z_m' \) on the lower half plane and \( z_1'', \ldots, z_n'' \) on the upper half plane. Without loss of generality, we assume that \( R \) is chosen large enough so that all the poles of \( f \) are inside the circle of radius \( R \) and centred at the origin. Suppose first of all that \( w \geq 0 \). Let \( C^-(R) \) denote the bottom half of the circle of radius \( R \) and centred at the origin, followed in the anticlockwise direction. Cauchy’s residue theorem then gives

\[
\frac{1}{2\pi i} \int_{C^-(R)} f(z)e^{-iwz} \, dz = \frac{1}{2\pi i} \int_{L(R)} f(z)e^{-iwz} \, dz = \sum_{j=1}^{m} \text{res}(f(z)e^{-iwz}, z_j').
\]

Note that if \( z = x + iy \in C^-(R) \), where \( x, y \in \mathbb{R} \), then \( y < 0 \) and \( |e^{-iwz}| = e^{wy} \leq 1 \). On the other hand, the length of the semicircle \( C^-(R) \) is equal to \( \pi R \). The condition (7.1) then guarantees that the integral over \( C^-(R) \) vanishes as \( R \to \infty \). Thus

\[
F(w) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{L(R)} f(z)e^{-iwz} \, dz = -i \sum_{j=1}^{m} \text{res}(f(z)e^{-iwz}, z_j').
\]

Suppose next that \( w \leq 0 \). Let \( C^+(R) \) denote the top half of the circle of radius \( R \) and centred at the origin, followed in the anticlockwise direction. Cauchy’s residue theorem then gives

\[
\frac{1}{2\pi i} \int_{C^+(R)} f(z)e^{-iwz} \, dz + \frac{1}{2\pi i} \int_{L(R)} f(z)e^{-iwz} \, dz = \sum_{j=1}^{n} \text{res}(f(z)e^{-iwz}, z_j'').
\]

Note that if \( z = x + iy \in C^+(R) \), where \( x, y \in \mathbb{R} \), then \( y > 0 \) and \( |e^{-iwz}| = e^{wy} \leq 1 \). On the other hand, the length of the semicircle \( C^+(R) \) is greater than \( \pi R \). The condition (7.1) then guarantees that the integral over \( C^+(R) \) vanishes as \( R \to \infty \). Thus

\[
F(w) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{L(R)} f(z)e^{-iwz} \, dz = i \sum_{j=1}^{n} \text{res}(f(z)e^{-iwz}, z_j'').
\]
Note that if \( z = x + iy \in C^+(R) \), where \( x, y \in \mathbb{R} \), then \( y > 0 \) and \( |e^{-iwz}| = e^{wy} \leq 1 \). On the other hand, the length of the semicircle \( C^+(R) \) is equal to \( \pi R \). The condition (7.1) then guarantees that the integral over \( C^+(R) \) vanishes as \( R \to \infty \). Thus

\[
F(w) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{L(R)} f(z)e^{-izw} \, dz = i \sum_{j=1}^{n} \text{res}(f(z)e^{-izw}, z_j').
\]

This completes the proof. \( \square \)

**Example 7.1.2.** Consider the function \( f : \mathbb{R} \to \mathbb{C} \), given for every \( x \in \mathbb{R} \) by

\[
f(x) = \frac{1}{x^2 + 1},
\]

Note that \( f(z) \) has simple poles at \( z = \pm i \). It follows from Theorem 7.1 that for every \( w \geq 0 \), we have

\[
F(w) = -i \text{ res} \left( \frac{e^{-iwz}}{z^2 + 1}, -i \right) = \frac{1}{2} e^{-w},
\]

and that for every \( w \leq 0 \), we have

\[
F(w) = i \text{ res} \left( \frac{e^{-iwz}}{z^2 + 1}, i \right) = \frac{1}{2} e^w.
\]

It follows that \( F(w) = \frac{1}{2} e^{-|w|} \) for every \( w \in \mathbb{R} \).

**Remark.** The condition (7.1) can be relaxed somewhat. It can be shown that a condition like

\[
\lim_{R \to \infty} \max_{|z| = R} |f(z)| = 0
\]

is sufficient. The proof of this assertion depends on Jordan’s lemma – the interested reader is referred to Chapter 11 of *Introduction to Complex Analysis*.

### 7.2. Application to the Heat Equation

The material in this section and the next will be unfamiliar to anyone who has not studied partial differential equations. As our aim is simply to illustrate the use of Fourier transforms in the study of partial differential equations, this unfamiliarity does not pose any serious difficulties.

In this section, we study a variant of the problem discussed in Section 5.4. We are interested in determining the solutions \( u = u(x, t) \) of a partial differential equation with boundary conditions, given by

\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0, & \text{if } -\infty < x < \infty \text{ and } 0 < t < \infty, \\
u(x, 0) = f(x), & \text{if } -\infty < x < \infty.
\end{cases}
\end{aligned}
\]

Here \( f \in G(\mathbb{R}) \), and \( k > 0 \) is a constant. This partial differential equation describes the temperature \( u = u(x, t) \) of an infinite rod, where the variables \( x \) and \( t \) represent position and time respectively. The second condition in (7.2) shows that the initial temperature distribution is given by the function \( f \). We also assume that the functions \( u \) and its derivatives discussed here are all piecewise continuous as a function of \( x \) for every fixed \( t \).

The approach is to pass over to the Fourier transform of \( u \) with respect to the variable \( x \), and then make use of Theorem 6.7 on the Fourier transforms of derivatives. More precisely, for every fixed \( t > 0 \), we consider the Fourier transform

\[
U(w, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, t)e^{-ixw} \, dx.
\]

Differentiating with respect to \( t \), we obtain

\[
U_t(w, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x, t)e^{-ixw} \, dx = -\frac{k}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x, t)e^{-ixw} \, dx = -\frac{kw^2}{2\pi} \int_{-\infty}^{\infty} u(x, t)e^{-ixw} \, dx,
\]

where the last equality is justified by Theorem 6.7. For each fixed \( w \in \mathbb{R} \), this leads to an ordinary differential equation of the type

\[
\frac{dU}{dt} + kw^2 U = 0,
\]

with general solution \( U(w, t) = A(w)e^{-kw^2t} \).
It is easy to see that $A(w) = U(w, 0) = F(w)$, so that

$$U(w, t) = F(w)e^{-kw^2t}.$$  \hfill (7.3)

Strictly speaking, we can now determine $F(w)$ and then apply the inverse Fourier transform on (7.3) to obtain $u$. However, an alternative way is to find a function $p(x, t)$ with Fourier transform $P(w, t) = e^{-kw^2t}$. Then $U(w, t) = F(w)P(w, t)$, and an application of Theorem 6.14 on convolution gives $2\pi u = f \ast p$ with respect to the variable $x$. More precisely, we have

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)p(x - y, t)\,dy.$$  \hfill (7.4)

To find the function $p(x, t)$, we appeal first to Example 6.1.3. The Fourier transform of the function $g(x) = e^{-x^2}$ is given by

$$G(w) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}w^2}.$$  

Suppose that $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Let $h(x) = \beta g(\alpha x)$ for every $x \in \mathbb{R}$. Then $h(x) = \beta e^{-\alpha^2x^2}$, and it follows from Theorems 6.3 and 6.4 that

$$H(w) = \frac{\beta}{|\alpha|} G\left(\frac{w}{\alpha}\right) = \frac{\beta}{2\sqrt{\pi|\alpha|}} e^{-w^2/4\alpha^2}.$$  

We now choose $\alpha$ and $\beta$ so that

$$\frac{\beta}{2\sqrt{\pi|\alpha|}} e^{-w^2/4\alpha^2} = e^{-kw^2t}, \quad \text{if } w \in \mathbb{R}.$$  

It is clear that we need

$$\alpha = \frac{1}{2\sqrt{kt}} \quad \text{and} \quad \beta = \sqrt{\frac{\pi}{kt}}.$$  

This gives

$$p(x, t) = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}.$$  

It now follows from (7.4) that the solution to the partial differential equation (7.2) is given by

$$u(x, t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(y)e^{-\frac{1}{4k}y^2}dy.$$  

### 7.3. Application to Laplace’s Equation

We are interested in determining the solutions $u = u(x, y)$ of Laplace’s equation with boundary conditions, given by

$$\begin{cases}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{if } -\infty < x < \infty \text{ and } 0 < y < \infty, \\
u(x, 0) = f(x), \quad \text{if } -\infty < x < \infty.
\end{cases}$$  \hfill (7.5)

Here $f \in G(\mathbb{R})$.

Again, we pass over to the Fourier transform of $u$ with respect to the variable $x$, and then make use of Theorem 6.7 on the Fourier transforms of derivatives. More precisely, for every fixed $y > 0$, we consider the Fourier transform

$$U(w, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x, y)e^{-ixw}\,dx.$$  

Differentiating twice with respect to $y$, we obtain

$$U_{yy}(w, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2}(x, y)e^{-ixw}\,dx = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x, t)e^{-ixw}\,dx = \frac{w^2}{2\pi} \int_{-\infty}^{\infty} u(x, t)e^{-ixw}\,dx,$$

where the last equality is justified by Theorem 6.7. For each fixed $w \in \mathbb{R}$, this leads to an ordinary differential equation of the type

$$\frac{d^2 U}{dy^2} - w^2 U = 0, \quad \text{with general solution } \quad U(w, y) = A(w)e^{iyw} + B(w)e^{-iyw}.$$
If we impose a boundedness condition on our solution, then this leads to \( A(w) = 0 \), and we omit the details here. In any case, this condition gives
\[
U(w, y) = B(w) e^{-|w|y}.
\]
It is easy to see that \( B(w) = U(w, 0) = F(w) \), so that
\[
U(w, y) = F(w) e^{-|w|y}.
\]
We now find a function \( p(x, y) \) with Fourier transform \( P(w, y) = e^{-|w|y} \). Then
\[
U(w, y) = F(w) P(w, y),
\]
and an application of Theorem 6.14 on convolution gives \( 2\pi u = f * p \) with respect to the variable \( x \).

More precisely, we have
\[
(7.6) \quad u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) p(x - t, y) \, dt.
\]
To find the function \( p(x, y) \), we appeal first to Example 7.1.2. The Fourier transform of the function \( g(x) = 1/(x^2 + 1) \) is given by
\[
G(w) = \frac{1}{2} e^{-|w|}.
\]
Suppose that \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{C} \). Let \( h(x) = \beta g(\alpha x) \) for every \( x \in \mathbb{R} \). Then \( h(x) = \beta / (\alpha^2 x^2 + 1) \), and it follows from Theorems 6.3 and 6.4 that
\[
H(w) = \frac{\beta}{|\alpha|} G\left(\frac{w}{\alpha}\right) = \frac{\beta}{2|\alpha|} e^{-|w/\alpha|}.
\]
We now choose \( \alpha \) and \( \beta \) so that
\[
\frac{\beta}{2|\alpha|} e^{-|w/\alpha|} = e^{-|w|y}, \quad \text{if } w \in \mathbb{R}.
\]
It is clear that we need
\[
\alpha = \frac{1}{y} \quad \text{and} \quad \beta = \frac{2}{y}.
\]
This gives
\[
p(x, y) = \frac{2y}{x^2 + y^2}.
\]
It now follows from (7.6) that the solution to the partial differential equation (7.2) is given by
\[
u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + y^2} \, dt.
\]
Problems for Chapter 7

1. Use Theorem 7.1 to compute the Fourier transform of each of the following functions \( f \in G(\mathbb{R}) \), where \( a, b \in \mathbb{R} \) are fixed and non-zero:

   (i) \( f(x) = \frac{1}{x^6 + 1} \)

   (ii) \( f(x) = \frac{x^2 + 1}{x^4 + 1} \)

   (iii) \( f(x) = \frac{x}{(x^2 + a^2)(x^2 + b^2)} \)

   (iv) \( f(x) = \frac{1}{(x - a)^2 + b^2} \)

2. Use your results in Problem 1 to evaluate each of the following integrals:

   (i) \( \int_0^\infty \frac{(x^2 + 1) \cos 6x}{x^4 + 1} \, dx \)

   (ii) \( \int_{-\infty}^\infty \frac{x \sin 3x}{(x^2 + a^2)(x^2 + b^2)} \, dx \)

3. Determine the solutions \( u = u(x, t) \) of a partial differential equation with boundary conditions, given by

\[
\begin{cases}
\frac{\partial u}{\partial t} - 4 \frac{\partial^2 u}{\partial x^2} = 0, & \text{if } -\infty < x < \infty \text{ and } 0 < t < \infty, \\
\frac{\partial u}{\partial x} = f(x), & \text{if } -\infty < x < \infty,
\end{cases}
\]

where

\[
f(x) = \begin{cases}
1, & \text{if } |x| \leq 2, \\
0, & \text{if } |x| > 2.
\end{cases}
\]