CHAPTER 5

Further Topics on Fourier Series

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5.1. Gibbs Phenomenon

As before, $E$ denotes the collection of all functions $f : [-\pi, \pi] \rightarrow \mathbb{C}$ which are piecewise continuous
on the interval $[-\pi, \pi]$. Each of these functions has at most a finite number of points of discontinuity,
at each of which the function need not be defined but must have one sided limits which are finite.
Again we adopt the convention that any two functions $f, g \in E$ are considered equal if $f(x) = g(x)$
for every $x \in [-\pi, \pi]$ with at most a finite number of exceptions. We may then assume, without
loss of generality, that any function $f \in E$ may be extended to a $2\pi$-periodic function defined on the
whole real line $\mathbb{R}$.

Recall that the Fourier series of a function $f \in E$ does not converge uniformly in any closed interval
which contains a point of discontinuity of $f$. The purpose of this section is to investigate the behaviour
of the Fourier series at such points of discontinuity. We begin by studying two familiar examples.

Example 5.1.1. Consider the function $f : [-\pi, \pi] \rightarrow \mathbb{C}$, given by

$$f(x) = \begin{cases} +1, & \text{if } 0 < x < \pi, \\ 0, & \text{if } x = 0 \text{ or } x = \pm\pi, \\ -1, & \text{if } -\pi < x < 0. \end{cases}$$

We have shown in Example 3.1.3 that this function has Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi n} \sin nx.$$

Furthermore, the Fourier series converges pointwise to $f(x)$ for every $x \in [-\pi, \pi]$ by Dirichlet’s
theorem. See also Example 1.2.4 for graphs of the partial sums $S_m(x)$.

We may assume, without loss of generality, that $m$ is even. Let $x_m = \pi/m$ and study the sum

$$S_m(x_m) = \sum_{n=1}^{m} \frac{4}{\pi n} \sin \frac{n\pi}{m} = 2 \sum_{n=1}^{m} \frac{2}{m} \sin \frac{(n\pi/m)}{m}.$$

This is a Riemann sum for the integral

$$2 \int_0^1 \frac{\sin \pi x}{\pi x} \, dx = \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} \, du \approx 1.18,$$

if we consider the dissection

$$0 < \frac{2}{m} < \frac{4}{m} < \ldots < \frac{m-2}{m} < 1.$$
We have shown in example hadad that this function has theier series of the Fourier series of theorema. See also example hadaeaf for graphs of the partial sums

We have and evaluate the function at the right hand endpoint of each subinterval. It follows that as $m \to \infty$, we have

$$S_m(x_m) \to \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} \, du \approx 1.18.$$ 

On the other hand, as $m \to \infty$, we have $x_m \to 0^+$ and so $f(x_m) \to 1$. Since $f(0+) = 1$ and $f(0-) = -1$, we conclude that

$$\lim_{m \to \infty} \frac{S_m(x_m) - f(x_m)}{f(0+) - f(0-)} \approx 0.09 \geq 0.0895.$$ 

**Example 5.1.2.** Consider the function $\phi : [-\pi, \pi] \to \mathbb{C}$, given by

$$\phi(x) = \begin{cases} x, & \text{if } -\pi < x < \pi, \\ 0, & \text{if } x = \pm \pi. \end{cases}$$

We have shown in Example 3.1.1 that this function has Fourier series

$$\phi(x) \sim \sum_{n=1}^\infty \frac{2(-1)^{n+1}}{n} \sin nx.$$ 

Furthermore, the Fourier series converges pointwise to $\phi(x)$ for every $x \in [-\pi, \pi]$ by Dirichlet’s theorem. See also Example 1.2.3 for graphs of the partial sums

$$T_m(x) = \sum_{n=1}^m \frac{2(-1)^{n+1}}{n} \sin nx.$$ 

Let $y_m = \pi - \pi/m$ and study the sum

$$T_m(y_m) = \sum_{n=1}^m \frac{2(-1)^{n+1}}{n} \sin \left(\pi - \frac{\pi}{m}\right) = \sum_{n=1}^m \frac{2}{n} \sin \frac{n\pi}{m} = 2\pi \sum_{n=1}^m \frac{1}{n} \sin \left(\frac{n\pi}{m}\right).$$

This is a Riemann sum for the integral

$$2\pi \int_0^1 \frac{\sin \pi y}{\pi y} \, dy = 2 \int_0^\pi \frac{\sin u}{u} \, du \approx 1.18\pi,$$ 

if we consider the dissection

$$0 < \frac{1}{m} < \frac{2}{m} < \ldots < \frac{m-1}{m} < 1$$

and evaluate the function at the midpoint of each subinterval. It follows that as $m \to \infty$, we have

$$T_m(y_m) \to 2\int_0^\pi \frac{\sin u}{u} \, du \approx 1.18\pi.$$ 

On the other hand, as $m \to \infty$, we have $y_m \to \pi^-$ and so $\phi(y_m) \to \pi$. Since $\phi(\pi) = -\pi$ and $\phi(\pi^-) = \pi$, we conclude that

$$\lim_{m \to \infty} \frac{T_m(y_m) - \phi(y_m)}{\phi(\pi) - \phi(\pi^-)} \approx -0.09 \leq -0.0895.$$ 

The two examples above illustrate the Gibbs phenomenon.

**Theorem 5.1.** Suppose that $f, f' \in E$, and that $d \in [-\pi, \pi]$ is a point of discontinuity of $f$. Then for the partial sums

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^m (a_n \cos nx + b_n \sin nx)$$

of the Fourier series of $f$, the following hold:

(i) There exists a sequence $x_1, x_2, x_3, \ldots$, satisfying $x_m > d$ for every $m \in \mathbb{N}$ and $x_m \to d$ as $m \to \infty$, such that

$$\lim_{m \to \infty} \frac{S_m(x_m) - f(x_m)}{f(d^+) - f(d^-)} \geq 0.089.$$
(ii) There exists a sequence \( y_1, y_2, y_3, \ldots \), satisfying \( y_m < d \) for every \( m \in \mathbb{N} \) and \( y_m \to d \) as \( m \to \infty \), such that

\[
\lim_{m \to \infty} \frac{S_m(y_m) - f(y_m)}{f(d + 0) - f(d - 0)} \leq -0.089.
\]

**Proof.** For simplicity, we shall assume that \( -\pi < d < \pi \). As in the proof of Theorem 4.7, we let \( j = f(d + 0) - f(d - 0) \) denote the jump of \( f \) at the discontinuity \( d \). Then the function

\[
g(x) = f(x) + \frac{j}{2\pi} \phi(x + \pi - d)
\]

is continuous at \( d \). Corresponding to

\[
f(x) = g(x) - \frac{j}{2\pi} \phi(x + \pi - d),
\]

we have

\[
S_m(x) = U_m(x) - \frac{j}{2\pi} T_m(x + \pi - d),
\]

where \( S_m, U_m \) and \( T_m \) denote respectively the partial sums of the Fourier series of \( f, g \) and \( \phi \). Clearly \( g, g' \in E \), and this means that \( g \) is continuous in some closed interval \( [a, b] \) such that \( d \in (a, b) \), and it follows from Theorem 4.7 that the Fourier series of \( g \) converges uniformly on \( [a, b] \). In particular, there exists \( N_1 \) such that

\[
|U_m(x) - g(x)| < 0.0005|j|,
\]

if \( m \geq N_1 \) and \( x \in [a, b] \).

On the other hand, it follows from Example 5.1.2 that there exists a sequence \( y_1, y_2, y_3, \ldots \), satisfying \( y_m < d \) for every \( m \in \mathbb{N} \) and \( y_m \to d \) as \( m \to \infty \), such that

\[
\lim_{m \to \infty} \frac{T_m(y_m + \pi - d) - \phi(y_m + \pi - d)}{2\pi} \approx 0.09.
\]

Hence there exists \( N_2 \) such that for every \( m \geq N_2 \), we have

\[
y_m \in [a, d) \quad \text{and} \quad \frac{T_m(y_m + \pi - d) - \phi(y_m + \pi - d)}{2\pi} \geq 0.0895.
\]

It now follows that for every \( m \geq \max\{N_1, N_2\} \), we have

\[
\begin{align*}
\left| \frac{S_m(y_m) - f(y_m)}{j} \right| & = \frac{U_m(y_m) - g(y_m)}{j} - \frac{T_m(y_m + \pi - d) - \phi(y_m + \pi - d)}{2\pi} \\
& \leq \frac{|U_m(y_m) - g(y_m)|}{|j|} - \frac{T_m(y_m + \pi - d) - \phi(y_m + \pi - d)}{2\pi} \\
& \leq 0.0005 - 0.0895 = -0.089.
\end{align*}
\]

This gives (ii). The proof of (i) is essentially similar. \( \Box \)

### 5.2. Differentiation and Integration

Suppose that \( f, f' \in E \), and that the Fourier series of \( f \) is given by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).
\]

The purpose of this section is to study conditions under which this can be differentiated term by term so that the Fourier series of \( f' \) is given by

\[
f'(x) \sim \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx).
\]

The following example demonstrates that extra conditions on the functions \( f \in E \) are required.

**Example 5.2.1.** Consider the function \( f : [-\pi, \pi] \to \mathbb{C} \), given by

\[
f(x) = \begin{cases} 
+1, & \text{if } 0 < x < \pi, \\
0, & \text{if } x = 0 \text{ or } x = \pm \pi, \\
-1, & \text{if } -\pi < x < 0.
\end{cases}
\]
We have shown in Example 3.1.3 that this function has Fourier series
\[ f(x) \sim \sum_{n=1}^{\infty} \frac{4}{\pi n} \sin nx. \]
Differentiating the right hand side term by term, we obtain the series
\[ \sum_{n=1}^{\infty} \frac{4}{\pi} \cos nx. \]
This is not a Fourier series of any function in \( E \), since the Riemann–Lebesgue lemma is not satisfied.
Quite clearly, we have \( f'(x) = 0 \) for every \( x \in [-\pi, \pi] \) except for \( x = 0 \) and \( x = \pm \pi \), so that the Fourier series is simply 0.

In fact, we have already established earlier some conditions under which differentiation term by term is justified. This is given in the proof of Theorem 4.6, where uniform convergence of Fourier series is studied. We state our result as follows.

**Theorem 5.2.** Suppose that \( f \in E \) is continuous on \([-\pi, \pi]\) and satisfies \( f(-\pi) = f(\pi) \). Suppose further that \( f' \in E \). If (5.1) is the Fourier series of \( f \), then the Fourier series of \( f' \) is given by (5.2), obtained from (5.1) by differentiating the Fourier series term by term.

We now turn to the question of integrating a Fourier series term by term. Suppose that we integrate the right hand side of (5.1) term by term. Then we obtain
\[ \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \left( \frac{-b_n}{n} \cos nx + \frac{a_n}{n} \sin nx \right), \]
clearly not a Fourier series. However, the coefficients now have an extra term \( n \) in the denominator, so we expect the series to converge at least.

**Theorem 5.3.** Suppose that \( f \in E \) has Fourier series (5.1). Then for every \( y \in [-\pi, \pi] \), we have
\[ \int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0 (y + \pi)}{2} + \sum_{n=1}^{\infty} \left( \frac{-b_n}{n} (\cos ny - \cos n\pi) + \frac{a_n}{n} \sin ny \right). \]
Furthermore, the series on the right hand side converges to the function on the left hand side uniformly on \([-\pi, \pi]\).

**Proof.** Note that if we blatantly integrate the two sides of (5.1) term by term over the interval \([-\pi, y]\), we obtain the two sides of (5.3). However, the right hand side of (5.3) is not a Fourier series, but we may write it in the form
\[ \frac{a_0 y}{2} + \left( \frac{a_0 \pi}{2} + \sum_{n=1}^{\infty} \frac{b_n \cos n\pi}{n} \right) + \sum_{n=1}^{\infty} \left( \frac{-b_n}{n} \cos ny + \frac{a_n}{n} \sin ny \right), \]
which looks like a Fourier series apart from the presence of the linear term on the left. Accordingly, for every \( y \in [-\pi, \pi] \), let
\[ g(y) = \int_{-\pi}^{\pi} f(x) \, dx - \frac{a_0 y}{2}. \]
Since \( f \in E \), the function \( g : [-\pi, \pi] \to \mathbb{C} \) defined in this way is continuous on \([-\pi, \pi]\), with
\[ g(-\pi) = \int_{-\pi}^{-\pi} f(x) \, dx + \frac{a_0 \pi}{2} = \frac{a_0 \pi}{2} \quad \text{and} \quad g(\pi) = \int_{-\pi}^{\pi} f(x) \, dx - \frac{a_0 \pi}{2} = \frac{a_0 \pi}{2}, \]
so that \( g(-\pi) = g(\pi) \). Furthermore, we have \( g'(y) = f(y) - \frac{1}{2} a_0 \) at each point of continuity of \( f \), and so \( g' \in E \). It follows from Theorem 4.6 that the Fourier series of \( g \) converges to \( g \) uniformly on \([-\pi, \pi]\). Suppose that the Fourier series of \( g \) is given by
\[ g(y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos ny + B_n \sin ny). \]
Note the equality in view of the uniform convergence of the Fourier series to \( g \). Then it follows from Theorem 5.2 that the Fourier series of \( g' \) is given by

\[
g'(y) \sim \sum_{n=1}^{\infty} (nB_n \cos ny - nA_n \sin ny).
\]

Combining (5.1) and (5.4), and noting that \( g'(y) = f(y) - \frac{1}{2}a_0 \) at each point of continuity of \( f \), we conclude that \( nB_n = a_n \) and \( -nA_n = b_n \) for every \( n \in \mathbb{N} \), thus giving

\[
\int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \left( \frac{b_n}{n} \cos ny + \frac{a_n}{n} \sin ny \right).
\]

Substituting \( y = -\pi \) gives

\[
0 = -\frac{a_0\pi}{2} + \frac{A_0}{2} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos n\pi, \quad \text{so that} \quad \frac{A_0}{2} = \frac{a_0\pi}{2} + \sum_{n=1}^{\infty} \frac{b_n}{n} \cos n\pi.
\]

This completes the proof of the theorem. \( \Box \)

### 5.3. Fourier Series on Other Intervals

For each closed interval \([a, b]\), where \( a, b \in \mathbb{R} \) and \( a < b \), we can consider the vector space \( E[a, b] \) of all functions \( f : [a, b] \rightarrow \mathbb{C} \) which are piecewise continuous on the interval \([a, b]\). We can then show that the sequence

\[
\left\{ \frac{1}{\sqrt{2}} \sin \frac{2\pi x}{b-a}, \cos \frac{2\pi x}{b-a}, \sin \frac{4\pi x}{b-a}, \cos \frac{4\pi x}{b-a}, \sin \frac{6\pi x}{b-a}, \cos \frac{6\pi x}{b-a}, \ldots \right\}
\]

in \( E[a, b] \) forms a closed infinite orthonormal system under the inner product given by

\[
\langle f, g \rangle = \frac{2}{b-a} \int_{a}^{b} f(x) \overline{g(x)} \, dx, \quad \text{if} \ f, g \in E[a, b].
\]

Then, for every \( f \in E[a, b] \), we can consider the Fourier series

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi n x}{b-a} + b_n \sin \frac{2\pi n x}{b-a} \right),
\]

where the Fourier coefficients are given by

\[
a_0 = \frac{2}{b-a} \int_{a}^{b} f(x) \, dx,
\]

and by

\[
a_n = \frac{2}{b-a} \int_{a}^{b} f(x) \cos \frac{2\pi n x}{b-a} \, dx \quad \text{and} \quad b_n = \frac{2}{b-a} \int_{a}^{b} f(x) \sin \frac{2\pi n x}{b-a} \, dx
\]

for every \( n \in \mathbb{N} \). However, all this work can be made redundant by a simple rescaling and periodicity argument which we now describe. Let \( E = E[-\pi, \pi] \).

The first step is a rescaling argument, and we consider intervals of the form \([-c, c]\), where \( c > 0 \) is fixed. Suppose that \( f \in E[-c, c] \). For every \( t \in [-\pi, \pi] \), we have \( ct/\pi \in [-c, c] \). Consider now the function \( g : [-\pi, \pi] \rightarrow \mathbb{C} \) given by

\[
g(t) = f \left( \frac{ct}{\pi} \right), \quad \text{if} \ t \in [-\pi, \pi].
\]

Clearly \( g \in E \), and so has Fourier series

\[
g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),
\]

where the Fourier coefficients are given by

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \, dt,
\]

and

\[
b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt \, dt.
\]

By the change of variables \( u = \frac{ct}{\pi} \), we have

\[
\int_{-\pi}^{\pi} f(t) \cos \frac{2\pi m x}{b-a} \, dt = \frac{2}{b-a} \int_{a}^{b} f(u) \cos \frac{2\pi m x}{b-a} \, du
\]

and

\[
\int_{-\pi}^{\pi} f(t) \sin \frac{2\pi m x}{b-a} \, dt = \frac{2}{b-a} \int_{a}^{b} f(u) \sin \frac{2\pi m x}{b-a} \, du
\]

for every \( m \in \mathbb{N} \). Therefore, the Fourier series of \( f \) on \([a, b]\) is given by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi n x}{b-a} + b_n \sin \frac{2\pi n x}{b-a}),
\]

and

\[
a_n = \frac{2}{b-a} \int_{a}^{b} f(x) \cos \frac{2\pi n x}{b-a} \, dx \quad \text{and} \quad b_n = \frac{2}{b-a} \int_{a}^{b} f(x) \sin \frac{2\pi n x}{b-a} \, dx
\]

for every \( n \in \mathbb{N} \). However, all this work can be made redundant by a simple rescaling and periodicity argument which we now describe. Let \( E = E[-\pi, \pi] \).
and by
\begin{equation}
(5.10) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt \, dt \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt \, dt
\end{equation}
for every $n \in \mathbb{N}$. Substituting $t = \frac{\pi x}{c}$, we deduce from (5.8)–(5.10) that
\begin{equation}
(5.11) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right),
\end{equation}
where the Fourier coefficients are given by
\begin{equation}
(5.12) \quad a_0 = \frac{1}{c} \int_{-c}^{c} f(x) \, dx,
\end{equation}
and by
\begin{equation}
(5.13) \quad a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} \, dx \quad \text{and} \quad b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} \, dx
\end{equation}
for every $n \in \mathbb{N}$.

The second step is a periodicity argument. Judiciously substituting $c = \frac{1}{2} (b - a)$, we deduce from (5.11)–(5.13) that
\begin{equation}
(5.14) \quad f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi x}{b - a} + b_n \sin \frac{2n\pi x}{b - a} \right),
\end{equation}
where the Fourier coefficients are given by
\begin{equation}
(5.15) \quad a_0 = \frac{2}{b - a} \int_{-c}^{c} f(x) \, dx,
\end{equation}
and by
\begin{equation}
(5.16) \quad a_n = \frac{2}{b - a} \int_{-c}^{c} f(x) \cos \frac{2n\pi x}{b - a} \, dx \quad \text{and} \quad b_n = \frac{2}{b - a} \int_{-c}^{c} f(x) \sin \frac{2n\pi x}{b - a} \, dx
\end{equation}
for every $n \in \mathbb{N}$. However, we may assume that the function $f \in E[a, b]$ is periodic with period $b - a = 2c$. Then all the integrands in (5.15) and (5.16) are periodic functions with period $b - a = 2c$, so that replacing the interval of integration $[-c, c]$ by the interval $[a, b]$ does not change the values of the integrals. With this observation, the results (5.5)–(5.7) become immediate consequences of (5.14)–(5.16).

We remark that the exponential version of our argument gives the following result. For every $f \in E[a, b]$, we have Fourier series
\begin{equation}
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i nx/(b-a)},
\end{equation}
where the Fourier coefficients are given by
\begin{equation}
c_n = \frac{1}{b - a} \int_{a}^{b} f(x) e^{-2\pi i nx/(b-a)} \, dx
\end{equation}
for every $n \in \mathbb{Z}$.

### 5.4. An Application to Partial Differential Equations

The material in this section will be unfamiliar to anyone who has not studied partial differential equations. As our aim is simply to illustrate the use of Fourier series in the study of partial differential equations, this unfamiliarity does not pose any serious difficulties.
We are interested in determining the solutions \( u = u(x,t) \) of a partial differential equation with boundary conditions, given by

\[
\begin{aligned}
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} &= 0, \quad \text{if } -L < x < L \text{ and } 0 < t < \infty, \\
u(x,0) &= f(x), \quad \text{if } -L \leq x \leq L, \\
u(-L,t) &= u(L,t), \quad \text{if } 0 \leq t < \infty, \\
\frac{\partial u}{\partial x}(-L,t) - \frac{\partial u}{\partial x}(L,t) &= 0, \quad \text{if } 0 \leq t < \infty.
\end{aligned}
\]

(5.17)

Here \( f : [-L,L] \to \mathbb{R} \) is a given continuous function satisfying \( f, f' \in E \) as well as \( f(-L) = f(L) \) and \( f'(-L) = f'(L) \), and \( k > 0 \) is a constant. This partial differential equation describes the temperature \( u = u(x,t) \) of a circular ring, where the variables \( x \) and \( t \) represent position and time respectively. The last two conditions in (5.17) are then simply an acknowledgment that the points \( x = -L \) and \( x = L \) represent one and the same endpoint of the ring. Finally, the second condition in (5.17) shows that the initial temperature distribution is given by the function \( f \).

The method we want to discuss is known as separation of variables, and we first find all non-trivial solutions of the form \( u(x,t) = X(x)T(t) \) of the system

\[
\begin{aligned}
\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} &= 0, \quad \text{if } -L < x < L \text{ and } 0 < t < \infty, \\
u(-L,t) &= u(L,t), \quad \text{if } 0 \leq t < \infty, \\
\frac{\partial u}{\partial x}(-L,t) - \frac{\partial u}{\partial x}(L,t) &= 0, \quad 0 \leq t < \infty;
\end{aligned}
\]

(5.18)

in other words, the system (5.17) without the information about the initial temperature distribution. Then

\[
\frac{\partial u}{\partial t} = X(x)T'(t) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t).
\]

Substituting these into the first equation of (5.18), we obtain

\[
X(x)T'(t) = kX''(x)T(t), \quad \text{or} \quad \frac{X''(x)}{X'(x)} = \frac{T'(t)}{kT(t)},
\]

assuming that we are not dividing by zero anywhere. Since the two variables \( x \) and \( t \) are independent of each other, the only possibility is that there is some unknown constant \( \lambda \) such that

\[
\frac{X''(x)}{X'(x)} = \frac{T'(t)}{kT(t)} = -\lambda
\]

(5.19)

for all values of \( x \) and \( t \). But the equations (5.19) give rise to two ordinary differential equations

\[
X''(x) + \lambda X(x) = 0 \quad \text{and} \quad T'(t) + k\lambda T(t) = 0.
\]

Let us now examine the last two conditions given in (5.18). With \( u(x,t) = X(x)T(t) \), the first of these becomes \( X(-L)T(t) = X(L)T(t) \) for every \( t \geq 0 \). The possibility \( T(t) = 0 \) for every \( t \geq 0 \) leads to an uninteresting trivial solution, so we assume that \( X(-L) = X(L) \). Similarly, the second of these two conditions leads to the assumption that \( X'(-L) = X'(L) \).

Let us first study the problem of the function \( X(x) \). We have

\[
\begin{aligned}
X''(x) + \lambda X(x) &= 0, \quad \text{if } -L < x < L, \\
X(-L) &= X(L), \\
X'(-L) &= X'(L).
\end{aligned}
\]

We shall take it as known that this system has non-trivial solutions precisely when \( \lambda \) is equal to one of the eigenvalues

\[
\lambda_n = \frac{n^2\pi^2}{L^2}, \quad \text{if } n = 0,1,2,3, \ldots.
\]

(5.20)

With \( \lambda_0 = 0 \), the equation becomes \( X''(x) = 0 \), with general solution \( X(x) = c_1 + c_2x \). The condition \( X(-L) = X(L) \) forces \( c_2 = 0 \), while the condition \( X'(-L) = X'(L) \) is satisfied for every choice of the coefficient \( c_1 \). Hence we have a solution \( X(x) = C \). For each \( n \geq 1 \), we have the equation

\[
X''(x) + \frac{n^2\pi^2}{L^2}X(x) = 0,
\]
with general solution

\[ X(x) = c_1 \cos \frac{n \pi x}{L} + c_2 \sin \frac{n \pi x}{L}. \]

and it is clear that the boundary conditions \( X(-L) = X(L) \) and \( X'(-L) = X'(L) \) are satisfied for every choice of the coefficients \( c_1 \) and \( c_2 \). In summary, leaving aside the coefficients, we have the solution \( X_0(x) = 1 \), as well as the solutions

\[ X^*_n(x) = \cos \frac{n \pi x}{L} \quad \text{and} \quad X^{**}_n(x) = \sin \frac{n \pi x}{L}, \]

for every \( n \in \mathbb{N} \).

Next, we study the problem of the function \( T(t) \). Clearly we need only consider the equation \( T''(t) + k \lambda T(t) = 0 \) when \( \lambda \) is equal to one of the eigenvalues (5.20). For each such eigenvalue \( \lambda_n \), it is easy to see that we have the non-trivial solution \( T_n(t) = e^{-k \lambda_n t} \). Combining the two parts of the problem, we now have the solution \( u_0(x, t) = X_0(x)T_0(t) = 1 \), as well as the solutions

\[ u^*_n(x, t) = X^*_n(x)T_n(t) = e^{-k \lambda_n t} \cos \frac{n \pi x}{L} \quad \text{and} \quad u^{**}_n(x, t) = X^{**}_n(x)T_n(t) = e^{-k \lambda_n t} \sin \frac{n \pi x}{L} \]

for every \( n \in \mathbb{N} \), leading to the general solution

\[ u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-k \lambda_n t} \left( a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \right). \]

Indeed, it can be proved that this is the general solution of the system (5.18), although the details are beyond the scope of these notes.

Let us now return to the system (5.17) and attempt to specialize the solution (5.21) to satisfy the condition concerning initial temperature distribution; in other words, the condition \( u(x, 0) = f(x) \) for every \( x \in [-L, L] \). Substituting \( t = 0 \) into the general solution (5.21), we obtain

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n \pi x}{L} + b_n \sin \frac{n \pi x}{L} \right). \]

Clearly the coefficients \( a_0, a_1, a_2, \ldots \) and \( b_1, b_2, b_3 \ldots \) are Fourier coefficients, and can be calculated by

\[ a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, \]

and by

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n \pi x}{L} \, dx \quad \text{and} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n \pi x}{L} \, dx \]

for every \( n \in \mathbb{N} \).
Problems for Chapter 5

1. Consider the function \( f : [-\pi, \pi] \to \mathbb{C} \), given by \( f(x) = |x| \) for every \( x \in [-\pi, \pi] \). Let
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
denote its Fourier series, and refer to Example 3.1.2 for the details.

   (i) Prove that the series
   \[
   \sum_{n=1}^{\infty} n a_n \sin nx
   \]
   converges for every \( x \in \mathbb{R} \), and sketch its graph on the interval \([-3\pi, 3\pi]\).

   (ii) Evaluate the sums
   \[
   \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.
   \]

2. Suppose that \( f \in E \) is continuous on \([-\pi, \pi]\), and that \( f(-\pi) = f(\pi) \) and \( f' \in E \). Let
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
denote its Fourier series.

   (i) Prove that \( na_n \to 0 \) and \( nb_n \to 0 \) as \( n \to \infty \).

   (ii) Is it necessarily true that \( n^2 a_n \to 0 \) and \( n^2 b_n \to 0 \) as \( n \to \infty \)? If so, give a proof. If not, give a counterexample.

3. Suppose that \( f \in E \) has Fourier series
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).
\]
Let \( c \in [-\pi, \pi] \) be fixed. For every \( y \in [-\pi, \pi] \), find a series expression for the integral
\[
\int_c^y f(x) \, dx
\]
in the spirit of Theorem 5.3, and show that the series converges to the integral uniformly on \([-\pi, \pi]\).

4. Consider the function \( f : \mathbb{R} \to \mathbb{C} \), given by \( f(x) = x \) for every \( x \in \mathbb{R} \). Let
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{and} \quad f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)
\]
denote respectively the Fourier series of \( f \) on \([-\pi, \pi]\) and on \([0, 2\pi]\), and let the function \( g : \mathbb{R} \to \mathbb{C} \) be defined for every \( x \in \mathbb{R} \) by
\[
g(x) = \frac{A_0 - a_0}{2} + \sum_{n=1}^{\infty} ((A_n - a_n) \cos nx + (B_n - b_n) \sin nx).
\]
Determine the precise value of \( g(x) \) for every \( x \in \mathbb{R} \), and justify every step of your argument.

5. Suppose that the function \( f : \mathbb{R} \to \mathbb{C} \) is piecewise continuous and periodic with period \( \pi \). Let
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{and} \quad f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos 2nx + B_n \sin 2nx)
\]
denote respectively the Fourier series of \( f \) on \([-\pi, \pi]\) and on \([0, \pi]\). Find relationships between the coefficients \( a_n, b_n, A_n \) and \( B_n \), and justify every step of your argument.

6. Suppose that \( f \in E \) is a periodic function defined on \( \mathbb{R} \) with period \( \pi \), and has Fourier series
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
on the interval $[-\pi, \pi]$. By studying the Fourier series of $f$ on the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, show that $a_n = b_n = 0$ for every odd $n \in \mathbb{N}$.

7. Discuss the problem of establishing a Parseval identity related to Fourier series for functions in $E[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. 