1. Let $\lambda$ be a real parameter.

(a) Determine the number of solutions $x = x(\lambda)$ of the equation $x^3 - 3x + 2\lambda = 0$ as a function of $\lambda$. (Hint: consider the graph of $x^3 - 3x$.)

(b) Sketch the graph of the curve $x^3 - 3x + 2\lambda = 0$ in the $(\lambda, x)$ plane.

(c) Determine the stability of the equilibrium points of the differential equation

$$\dot{x} + x^3 - 3x + 2\lambda = 0.$$ 

2. Locate the equilibrium points and sketch the phase diagram for the following system:

$$\begin{align*}
\dot{x} &= x - y \\
\dot{y} &= x + y - 2xy.
\end{align*}$$

3. A certain first order system of two differential equations is known to have exactly two equilibria, at $(1, 0)$ and $(-1, 0)$, both saddles. Draw a possible phase plane diagram assuming either

(a) no trajectory connects the saddles, or

(b) there is a trajectory (a separatrix) connecting the saddles.

4. Show that the following systems have no periodic solutions:

(a)

$$\begin{align*}
\dot{x} &= y + x^3 \\
\dot{y} &= x + y + y^3,
\end{align*}$$

(b)

$$\begin{align*}
\dot{x} &= y \\
\dot{y} &= -(1 + x^2 + x^4)y - x.
\end{align*}$$

5. Consider the system

$$\begin{align*}
\dot{x} &= X(x, y) \\
\dot{y} &= Y(x, y),
\end{align*}$$

in some simply connected region $R$ of the plane, where $X$ and $Y$ have continuous first partial derivatives. Suppose that in the region $R$ there exists a function $P(x, y)$ with continuous first partial derivatives such that

$$\frac{\partial}{\partial x} (PX) + \frac{\partial}{\partial y} (PY)$$

is always positive. Prove that the system (1) has no periodic solutions. (Use the ideas in the proof of Bendixon’s negative criterion.)