T(1) Theory for compact Calderón-Zygmund operators

K.M. Perfekt, S. Pott, P. Villarroya

University of Lund

AMSI/AustMS Workshop in Harmonic Analysis and its Applications

Sydney, 21 of July 2014
The $T(1)$ Theorem

The classical $T(1)$ Theorem gives necessary and sufficient conditions for boundedness on $L^p(\mathbb{R}^n)$ of non-convolution singular integral operators

$$T(f)(x) = \int_{\mathbb{R}^n} f(t)K(t, x)dt$$

with $K$ a standard Calderón-Zygmund kernel, $f$ smooth and compactly supported and $x \notin \text{supp}(f)$. 
The \textit{T}(1) \textit{Theorem} 

The classical \textit{T}(1) \textit{Theorem} gives necessary and sufficient conditions for boundedness on $L^p(\mathbb{R}^n)$ of non-convolution singular integral operators 

$$T(f)(x) = \int_{\mathbb{R}^n} f(t)K(t,x)dt$$

with $K$ a standard Calderón-Zygmund kernel, $f$ smooth and compactly supported and $x \notin \text{supp}(f)$.

**Definition.** $K : \mathbb{R}^{2n} \setminus \{x = t\} \rightarrow \mathbb{C}$ is a standard Calderón-Zygmund kernel if there exist $0 < \delta \leq 1$ such that

- $|K(t, x)| \lesssim \frac{1}{|t - x|^n}$
- $|K(t, x) - K(t', x')| \lesssim \frac{(|x - x'| + |t - t'|)^\delta}{|t - x|^{n+\delta}}$

whenever $2(|x - x'| + |t - t'|) \leq |t - x|$
Theorem (David-Journé). Let $T : S(\mathbb{R}^n) \to S(\mathbb{R}^n)'$ be a continuous linear operator with a standard Calderón-Zygmund kernel. Then, $T$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $T$ satisfies

- the weak boundedness condition $|\langle T(\varphi_I), \varphi_I \rangle| \lesssim C$
- the cancellation conditions $T(1), T^*(1) \in \text{BMO}(\mathbb{R}^n)$
Compact $T(1)$ theorem

The Compact $T(1)$ Theorem gives necessary and sufficient conditions for compactness on $L^p(\mathbb{R}^n)$ of non-convolution singular integral operators

$$T(f)(x) = \int_{\mathbb{R}^n} f(t)K(t, x)dt$$
Compact $T(1)$ theorem

The Compact $T(1)$ Theorem gives necessary and sufficient conditions for compactness on $L^p(\mathbb{R}^n)$ of non-convolution singular integral operators

$$T(f)(x) = \int_{\mathbb{R}^n} f(t)K(t,x)dt$$

Let $L, S, D : [0, \infty) \to \mathbb{R}$ be three bounded functions such that

$$\lim_{x \to \infty} L(x) = \lim_{x \to 0} S(x) = \lim_{x \to \infty} D(x) = 0$$
Compact $T(1)$ theorem

The Compact $T(1)$ Theorem gives necessary and sufficient conditions for compactness on $L^p(\mathbb{R}^n)$ of non-convolution singular integral operators

$$T(f)(x) = \int_{\mathbb{R}^n} f(t)K(t, x)dt$$

Let $L, S, D : [0, \infty) \to \mathbb{R}$ be three bounded functions such that

$$\lim_{x \to \infty} L(x) = \lim_{x \to 0} S(x) = \lim_{x \to \infty} D(x) = 0$$

Definition. A function $K : \mathbb{R}^{2n} \setminus \{x = y\} \to \mathbb{C}$ is a compact Calderón-Zygmund kernel if for some $0 < \delta \leq 1$

$$|K(t, x) - K(t', x')| \lesssim \frac{(|x - x'| + |t - t'|)^\delta}{|t - x|^{n+\delta}} L(|t - x|)S(|t - x|)D(|t + x|)$$

whenever $2(|x - x'| + |t - t'|) \leq |t - x|$. 
Definition. A linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$ is associated with a compact Calderón-Zygmund kernel $K$ if

$$\langle T(f), g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t)g(x)K(t, x) \, dt \, dx$$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with compact disjoint supports.
Definition. A linear operator \( T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)' \) is associated with a compact Calderón-Zygmund kernel \( K \) if

\[
\langle T(f), g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t)g(x)K(t, x) \, dt \, dx
\]

for all \( f, g \in \mathcal{S}(\mathbb{R}^n) \) with compact disjoint supports.

Notation. Given two cubes \( I, J \), we denote by \( \langle I, J \rangle \) the smallest cube containing \( I \cup J \) and we write its side length as \( \text{diam}(I, J) \).

We define the relative distance between \( I \) and \( J \) as

\[
\text{rdist}(I, J) = \frac{\text{diam}(I, J)}{\max(\ell(I), \ell(J))}
\]

and their eccentricity as

\[
\text{ec}(I, J) = \frac{\min(|I|, |J|)}{\max(|I|, |J|)}
\]

where \( |I|, \ell(I) \) denote the measure and side length of \( I \) respectively.

Finally, we write \( \mathbb{B} = [-1/2, 1/2]^n \) and \( \mathbb{B}_\lambda = \lambda \mathbb{B} \).
Definition. A bounded continuous function $\phi$ is adapted to a cube $I$ if for all $t, x \in \mathbb{R}^n$

\[
|\phi(x)| \leq C|I|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I)|}{\ell(I)}\right)^{-N}
\]

\[
|\phi(x) - \phi(t)| \leq C\left(\frac{|t - x|}{\ell(I)}\right)\alpha |I|^{-\frac{1}{2}} \sup_{r \in \langle t, x \rangle} \left(1 + \frac{|r - c(I)|}{\ell(I)}\right)^{-N}
\]

where $c(I)$ is the centre of $I$ and $\langle t, x \rangle$ denotes the smallest cube containing $\{t\} \cup \{x\}$. 
Definition. A bounded continuous function \( \phi \) is adapted to a cube \( I \) if for all \( t, x \in \mathbb{R}^n \)

\[
|\phi(x)| \leq C|I|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I)|}{\ell(I)}\right)^{-N}
\]

\[
|\phi(x) - \phi(t)| \leq C \left(\frac{|t - x|}{\ell(I)}\right)^{\alpha} |I|^{-\frac{1}{2}} \sup_{r \in \langle t, x \rangle} \left(1 + \frac{|r - c(I)|}{\ell(I)}\right)^{-N}
\]

where \( c(I) \) is the centre of \( I \) and \( \langle t, x \rangle \) denotes the smallest cube containing \( \{t\} \cup \{x\} \).

Definition. A linear operator \( T \) satisfies the weak compactness condition if there are functions \( L, S, D \) such that

\[
|\langle T(\phi_I), \varphi_I \rangle| \lesssim C L(\ell(I)) S(\ell(I)) D(\text{rdist}(I, \mathbb{B}))
\]

for any cube \( I \) and every pair \( \phi_I, \varphi_I \) of bump functions adapted to \( I \).

We write \( F(I, J, K) = L(\ell(I)) S(\ell(J)) D(\text{rdist}(K, \mathbb{B})) \) and \( F(I) = F(I, I, I) \).
Definition. For $M \in \mathbb{N}$, let $\mathcal{I}_M$ ($\mathcal{D}_M$) be the family of cubes (dyadic cubes) such that

$$2^{-M} \leq \ell(I) \leq 2^M, \quad \text{rdist}(I, \mathbb{B}_{2^M}) \leq M$$

We call any cube $I \in \mathcal{I}_M$ (or $I \in \mathcal{D}_M$) a lagom cube.
Definition. For $M \in \mathbb{N}$, let $\mathcal{I}_M$ ($\mathcal{D}_M$) be the family of cubes (dyadic cubes) such that

$$2^{-M} \leq \ell(I) \leq 2^M, \quad \text{rdist}(I, \mathbb{B}_{2^M}) \leq M$$

We call any cube $I \in \mathcal{I}_M$ (or $I \in \mathcal{D}_M$) a lagom cube.

Definition. Let $(\psi_I)_{I \in \mathcal{D}}$ be a wavelet basis of $L^2(\mathbb{R}^n)$. Then, for every $M \in \mathbb{N}$, we define the lagom projection operator $P_M$ by

$$P_M(f) = \sum_{I \in \mathcal{D}_M} \langle f, \psi_I \rangle \psi_I$$

We also define the orthogonal projection as $P_M(f)^\perp = f - P_M(f)$. 

Theorem. $T$ is compact if and only if $\lim_{M \to \infty} \| P_M \circ T \| = 0$. 

Definition. For $M \in \mathbb{N}$, let $\mathcal{I}_M$ ($\mathcal{D}_M$) be the family of cubes (dyadic cubes) such that
\[ 2^{-M} \leq \ell(I) \leq 2^M, \quad \text{rdist}(I, \mathbb{B}_{2^M}) \leq M \]
We call any cube $I \in \mathcal{I}_M$ (or $I \in \mathcal{D}_M$) a lagom cube.

Definition. Let $(\psi_I)_{I \in \mathcal{D}}$ be a wavelet basis of $L^2(\mathbb{R}^n)$. Then, for every $M \in \mathbb{N}$, we define the lagom projection operator $P_M$ by
\[ P_M(f) = \sum_{I \in \mathcal{D}_M} \langle f, \psi_I \rangle \psi_I \]
We also define the orthogonal projection as $P_M^\perp(f) = f - P_M(f)$.

Theorem. $T$ is compact if and only if
\[ \lim_{M \to \infty} \| P_M^\perp \circ T \| = 0 \]
**Definition.** We define \( \text{CMO}(\mathbb{R}^n) \) as the closure in \( \text{BMO}(\mathbb{R}^n) \) of continuous functions vanishing at infinity.
**Definition.** We define $\text{CMO}(\mathbb{R}^n)$ as the closure in $\text{BMO}(\mathbb{R}^n)$ of continuous functions vanishing at infinity.

**Lemma.** The following statements are equivalent:

- $f \in \text{CMO}(\mathbb{R}^n)$
- $f \in \text{BMO}(\mathbb{R}^n)$ and
  $$\lim_{M \to 0} \sup_{I \notin \mathcal{I}_M} \frac{1}{|I|} \int_I \left| f(x) - \frac{1}{|I|} \int_I f(y) dy \right| dx = 0$$
- $f \in \text{BMO}(\mathbb{R}^n)$ and
  $$\lim_{M \to \infty} \sup_{\Omega \subset \mathbb{R}^n} \left( \frac{1}{|\Omega|} \sum_{J \notin \mathcal{D}_M} |\langle f, \psi_J \rangle|^2 \right)^{\frac{1}{2}} = 0$$
Theorem. Let $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)'$ be a continuous linear operator with a standard Calderón-Zygmund kernel $K$.

Then, $T$ can be extended to a compact operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $K$ is a compact Calderón-Zygmund kernel and $T$ satisfies

- the weak compactness condition $|\langle T(\varphi_I), \varphi_I \rangle| \lesssim CF(I)$
- the cancellation conditions $T(1), T^*(1) \in \text{CMO}(\mathbb{R}^n)$

The hypotheses are also equivalent to compactness at the endpoints:

- from $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$
- from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$
- from $L^\infty(\mathbb{R}^n)$ into $\text{CMO}(\mathbb{R}^n)$

With the extra conditions $T(1) = T^*(1) = 0$, we obtain compactness on $H^1(\mathbb{R}^n)$ and on $\text{CMO}(\mathbb{R}^n)$. 
Theorem. Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)'$ be a continuous linear operator with a standard Calderón-Zygmund kernel $K$.

Then, $T$ can be extended to a compact operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $K$ is a compact Calderón-Zygmund kernel and $T$ satisfies

- the weak compactness condition $|\langle T(\phi_I), \varphi_I \rangle| \lesssim C F(I)$
- the cancellation conditions $T(1), T^*(1) \in \text{CMO}(\mathbb{R}^n)$

The hypotheses are also equivalent to compactness at the endpoints:

- from $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$
- from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$
- from $L^\infty(\mathbb{R}^n)$ into $\text{CMO}(\mathbb{R}^n)$

With the extra conditions $T(1) = T^*(1) = 0$, we obtain compactness on $H^1(\mathbb{R}^n)$ and on $\text{CMO}(\mathbb{R}^n)$. 
Proposition (Cauchy Integral). Let $A : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with $A' \in L^\infty \cap CMO$.

Let $\Gamma \subset \mathbb{C}$ be a curve parametrized by $\Gamma \equiv \{x + iA(x) : x \in \mathbb{R}\}$ and

$$H_\Gamma f(z) = \int_\Gamma \frac{f(w)}{\text{Re}(z - w)} \, ds(w)$$

Let $T_\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ be the operator defined by

$$T_\Gamma f(z) = \int_\Gamma \frac{f(w)}{z - w + 2(\sigma_{z,w} - \tau_{z,w})} \, ds(w)$$

with $ds$ the arc length measure on $\Gamma$, $z = x + iA(x)$, $w = t + iA(t)$, and

$$\sigma_{z,w} = x - (x - t)/4 + iA(x - (x - t)/4)$$

$$\tau_{z,w} = x - 3(x - t)/4 + iA(x - 3(x - t)/4)$$

Then, $T_\Gamma - H_\Gamma$ is compact on $L^p(\Gamma)$. 
Necessity of the hypotheses

Lemma. Let $T$ be a linear operator with a standard Calderón-Zygmund kernel $K$.
Let $1 < p < \infty$. If $T$ extends compactly on $L^p(\mathbb{R}^n)$ then, $K$ is a compact Calderón-Zygmund kernel.
Necessity of the hypotheses

Lemma. Let $T$ be a linear operator with a standard Calderón-Zygmund kernel $K$.
Let $1 < p < \infty$. If $T$ extends compactly on $L^p(\mathbb{R}^n)$ then, $K$ is a compact Calderón-Zygmund kernel.

Proof. We need to show that there exist $0 < \delta' \leq 1$ such that

$$C(t, x) = \frac{|t - x|^{n+\delta'}}{|t - t'|^{\delta'}} |K(t, x) - K(t', x)|$$

with fixed arbitrary $t'$ so that $2|t - t'| < |t - x|$, tends to zero when $|t - x| \to \infty$; or $|t - x| \to 0$; or $|t + x| \to \infty$ while $|t - x|$ is bounded above and below.
Necessity of the hypotheses

**Lemma.** Let $T$ be a linear operator with a standard Calderón-Zygmund kernel $K$.
Let $1 < p < \infty$. If $T$ extends compactly on $L^p(\mathbb{R}^n)$ then, $K$ is a compact Calderón-Zygmund kernel.

**Proof.** We need to show that there exist $0 < \delta' \leq 1$ such that

$$C(t, x) = \frac{|t - x|^{n+\delta'}}{|t - t'|^{\delta'}} |K(t, x) - K(t', x)|$$

with fixed arbitrary $t'$ so that $2|t - t'| < |t - x|$, tends to zero when $|t - x| \to \infty$; or $|t - x| \to 0$; or $|t + x| \to \infty$ while $|t - x|$ is bounded above and below.

Since $K$ is a standard Calderón-Zygmund kernel,

$$C(t, x) \leq C \frac{|t - t'|^{\delta - \delta'}}{|t - x|^{\delta - \delta'}}$$

for $2|t - t'| < |t - x|$ and any $0 < \delta' < \delta$. 
Therefore, we can assume $|t - t'| \approx |t - x|$ and so,

$$C(t, x) \lesssim |t - x|^n |K(t, x) - K(t', x)|$$
Therefore, we can assume $|t - t'| \approx |t - x|$ and so,

$$C(t, x) \lesssim |t - x|^n |K(t, x) - K(t', x)|$$

From the kernel properties, we have that for $\epsilon > 0$ and $0 < \lambda < \epsilon^{1/\delta} |t - t'|$,

$$\left| \int \int T_t D^1_\lambda \Phi(u) T_x D^1_\lambda \Phi(y) K(u, y) dudy - K(t, x) \right| < \epsilon \frac{|t - t'|^\delta}{|t - x|^{n+\delta}}$$
Therefore, we can assume $|t - t'| \approx |t - x|$ and so,

$$C(t, x) \lesssim |t - x|^n |K(t, x) - K(t', x)|$$

From the kernel properties, we have that for $\epsilon > 0$ and $0 < \lambda < \epsilon^{1/\delta}|t - t'|$,

$$\left| \int \int T_t D^1_{\lambda} \Phi(u) T_x D^1_{\lambda} \Phi(y) K(u, y) dudy - K(t, x) \right| < \epsilon \frac{|t - t'|^{\delta}}{|t - x|^{n+\delta}}$$

Then, taking appropriate sequences, we have

$$C(t_m, x_m) \lesssim |t_m - x_m|^n \left| \int \int T_{t_m} D^1_{\lambda} \Phi(u) T_{x_m} D^1_{\lambda} \Phi(y) K(u, y) dudy \
- \int \int T_{t_m} D^1_{\lambda} \Phi(u) T_{x_m} D^1_{\lambda} \Phi(y) K(u, y) dudy \right|$$

$$= |t_m - x_m|^n \left| \langle T(T_{t_m} D^1_{\lambda} \Phi) - T(T_{t'_m} D^1_{\lambda} \Phi), T_{x_m} D^1_{\lambda} \Phi \rangle \right|$$

$$= \left| \langle T(f_m), g_m \rangle \right|$$
Since $\|f_m\|_{L^p(\mathbb{R}^n)} \leq C$, compactness of $T$ guarantees the existence of a convergent subsequence $(T(f_{m_k}))_{k \in \mathbb{N}}$ and so, $\|T(f_{m_k}) - T(f_{m_l})\|_{L^p(\mathbb{R}^n)} < \tilde{\epsilon}$.

With this,

$$C(t_{m_k}, x_{m_k}) \lesssim |\langle T(f_{m_k}), g_{m_k} \rangle|$$

$$\leq |\langle T(f_{m_k}) - T(f_{m_l}), g_{m_k} \rangle| + |\langle T(f_{m_l}), g_{m_k} \rangle|$$

and every term is proven to tend to zero.
Since $\|f_m\|_{L^p(\mathbb{R}^n)} \leq C$, compactness of $T$ guarantees the existence of a convergent subsequence $(T(f_{m_k}))_{k \in \mathbb{N}}$ and so, $\|T(f_{m_k}) - T(f_{m_l})\|_{L^p(\mathbb{R}^n)} < \tilde{\epsilon}$.

With this,

$$C(t_{m_k}, x_{m_k}) \lesssim |\langle T(f_{m_k}), g_{m_k} \rangle|$$

$$\leq |\langle T(f_{m_k}) - T(f_{m_l}), g_{m_k} \rangle| + |\langle T(f_{m_l}), g_{m_k} \rangle|$$

and every term is proven to tend to zero.

**Lemma.** Let $T$ be a compact operator on $L^p(\mathbb{R}^n)$. Then, $T$ satisfies the weak compactness condition.
Since $\|f_m\|_{L^p(\mathbb{R}^n)} \leq C$, compactness of $T$ guarantees the existence of a convergent subsequence $(T(f_{m_k}))_{k \in \mathbb{N}}$ and so, $\|T(f_{m_k}) - T(f_{m_l})\|_{L^p(\mathbb{R}^n)} < \tilde{\epsilon}$.

With this, 

$$C(t_{m_k}, x_{m_k}) \lesssim |\langle T(f_{m_k}), g_{m_k} \rangle|$$

$$\leq |\langle T(f_{m_k}) - T(f_{m_l}), g_{m_k} \rangle| + |\langle T(f_{m_l}), g_{m_k} \rangle|$$

and every term is proven to tend to zero.

**Lemma.** Let $T$ be a compact operator on $L^p(\mathbb{R}^n)$. Then, $T$ satisfies the weak compactness condition.

For every $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for every cube $I \subset \mathbb{R}^n$ and every $\phi_I, \varphi_I$ bump functions adapted to $I$ with order $N$ and $C > 0$

$$|\langle T(\phi_I), \varphi_I \rangle| \leq C\left(\left(1 + \frac{\ell(I)}{2^M}\right)^{-\alpha} \left(1 + \frac{2^{-M}}{\ell(I)}\right)^{-\alpha} \left(\text{rdist}(I, B_{2M})\right)^{-N} + \epsilon\right)$$

with $\alpha = |1/2 - 1/p| + 1/2$. 
Let $0 \leq \Phi \leq 1$ with $\Phi(x) = 1$ for $|x| \leq 1$ and $\Phi(x) = 0$ for $|x| > 2$.

For $a \in \mathbb{R}^n$, $\lambda > 0$, we define the translation and dilation operators by $T_a f(x) = f(x - a)$ and $D_\lambda f(x) = f(x/\lambda)$ respectively.
Let $0 \leq \Phi \leq 1$ with $\Phi(x) = 1$ for $|x| \leq 1$ and $\Phi(x) = 0$ for $|x| > 2$.

For $a \in \mathbb{R}^n$, $\lambda > 0$, we define the translation and dilation operators by $T_a f(x) = f(x - a)$ and $D_\lambda f(x) = f(x/\lambda)$ respectively.

**Lemma.** Let $T$ be a linear operator with a compact Calderón-Zygmund kernel with parameter $\delta > 0$. Let $I \subset \mathbb{R}^n$ be a cube and $f \in S(\mathbb{R}^n)$ with compact support in $I$ and mean zero. Then, the limit

$$\langle T(1), f \rangle = \lim_{k \to \infty} \langle T(T_c(I) D_{2^k \ell(I)}(1) \Phi), f \rangle$$

exists. Moreover, we have the error bound

$$|\langle T(1), f \rangle - \langle T(T_c(I) D_{2^k \ell(I)}(1) \Phi), f \rangle| \lesssim 2^{-k\delta} F_K(2^k I) \|f\|_{L^1(\mathbb{R}^n)}$$
Let $0 \leq \Phi \leq 1$ with $\Phi(x) = 1$ for $|x| \leq 1$ and $\Phi(x) = 0$ for $|x| > 2$.

For $a \in \mathbb{R}^n$, $\lambda > 0$, we define the translation and dilation operators by $T_a f(x) = f(x - a)$ and $D_\lambda f(x) = f(x/\lambda)$ respectively.

**Lemma.** Let $T$ be a linear operator with a compact Calderón-Zygmund kernel with parameter $\delta > 0$. Let $I \subset \mathbb{R}^n$ be a cube and $f \in S(\mathbb{R}^n)$ with compact support in $I$ and mean zero. Then, the limit

$$\langle T(1), f \rangle = \lim_{k \to \infty} \langle T(T_c(I)D_{2^k\ell(I)}\Phi), f \rangle$$

exists. Moreover, we have the error bound

$$|\langle T(1), f \rangle - \langle T(T_c(I)D_{2^k\ell(I)}\Phi), f \rangle| \lesssim 2^{-k\delta} F_K(2^k I) \|f\|_{L^1(\mathbb{R}^n)}$$

**Lemma.** Let $T$ associated with a compact Calderón-Zygmund kernel and compact on $L^p(\mathbb{R}^n)$. Then $T(1), T^*(1) \in CMO(\mathbb{R}^n)$. 

Sufficiency of hypotheses. The special cancellation case

The Bump lemma. Let $T$ be an operator with compact Calderón-Zygmund kernel and parameter $\delta > 0$ satisfying the weak compactness condition and the special cancellation conditions $T(1) = 0, \ T^*(1) = 0$. 

Let $I, J$ two cubes and $\psi_I, \psi_J$ be bump functions compactly supported and adapted to $I$ and $J$ respectively with mean zero. Then,

$$|\langle T(\psi_I), \psi_J \rangle| \lesssim ec(I, J)^{1/2} + \delta n \text{dist}(I, J)^{- (n + \delta)}F(I, J)$$

where $F$ is a function such that when $\ell(J) \leq \ell(I)$:

$F(I, J) = F_K(\langle I, J \rangle, J, \langle I, J \rangle)$ when $\text{dist}(I, J) > 3$, 

$F(I, J) = \tilde{F}_K(I, J, I) + F_W(J) + F_K(J, J, I)$ when $\text{dist}(I, J) \leq 3$ and 

$\tilde{F}(I, J, K) = L(\ell(I)) S(\ell(J)) \sum_{j \geq 0} 2^{-j} D(2^j K, B)$
Sufficiency of hypotheses. The special cancellation case

The Bump lemma. Let $T$ be an operator with compact Calderón-Zygmund kernel and parameter $\delta > 0$ satisfying the weak compactness condition and the special cancellation conditions $T(1) = 0$, $T^*(1) = 0$. Let $I$, $J$ two cubes and $\psi_I$, $\psi_J$ be bump functions compactly supported and adapted to $I$ and $J$ respectively with mean zero.

\[
|\langle T(\psi_I), \psi_J \rangle| \lesssim e^{c(I,J)} \left( \frac{1}{2} + \delta \left( \frac{\text{rdist}(I,J)}{n} - (n + \delta) \right) F(I,J) \right)
\]

where $F$ is a function such that when $\ell(J) \leq \ell(I)$:

- $F(I,J) = F_K(\langle I,J \rangle, J, \langle I,J \rangle)$ when $\text{rdist}(I,J) > 3$,
- $F(I,J) = \tilde{F}_K(I,J) + F_W(J) + F_K(J,J,I)$ when $\text{rdist}(I,J) \leq 3$.

And $\tilde{F}(I,J,K) = L(\ell(I)) S(\ell(J)) \sum_{j \geq 0} 2^{-j} \delta D(2^j K, B)$. 
The Bump lemma. Let $T$ be an operator with compact Calderón-Zygmund kernel and parameter $\delta > 0$ satisfying the weak compactness condition and the special cancellation conditions $T(1) = 0$, $T^*(1) = 0$. Let $I, J$ two cubes and $\psi_I, \psi_J$ be bump functions compactly supported and adapted to $I$ and $J$ respectively with mean zero. Then,

$$|\langle T(\psi_I), \psi_J \rangle| \lesssim \text{ec}(I, J)^{\frac{1}{2} + \frac{\delta}{n}} \text{rdist}(I, J)^{-(n+\delta)} F(I, J)$$
Sufficiency of hypotheses. The special cancellation case

The Bump lemma. Let $T$ be an operator with compact Calderón-Zygmund kernel and parameter $\delta > 0$ satisfying the weak compactness condition and the special cancellation conditions $T(1) = 0$, $T^*(1) = 0$. Let $I, J$ two cubes and $\psi_I, \psi_J$ be bump functions compactly supported and adapted to $I$ and $J$ respectively with mean zero. Then,

$$|\langle T(\psi_I), \psi_J \rangle| \lesssim ec(I, J)^{\frac{1}{2} + \frac{\delta}{n}} \text{rdist}(I, J)^{-(n+\delta)} F(I, J)$$

where $F$ is a function such that when $\ell(J) \leq \ell(I)$:

- $F(I, J) = F_K(\langle I, J \rangle, J, \langle I, J \rangle)$ when $\text{rdist}(I, J) > 3,$
- $F(I, J) = \tilde{F}_K(I, J, I) + F_W(J) + F_K(J, J, I)$ when $\text{rdist}(I, J) \leq 3$
Sufficiency of hypotheses. The special cancellation case

The Bump lemma. Let $T$ be an operator with compact Calderón-Zygmund kernel and parameter $\delta > 0$ satisfying the weak compactness condition and the special cancellation conditions $T(1) = 0$, $T^*(1) = 0$. Let $I, J$ two cubes and $\psi_I, \psi_J$ be bump functions compactly supported and adapted to $I$ and $J$ respectively with mean zero. Then,

$$|\langle T(\psi_I), \psi_J \rangle| \lesssim ec(I, J)^{\frac{1}{2} + \frac{\delta}{n}} \text{rdist}(I, J)^{-(n+\delta)} F(I, J)$$

where $F$ is a function such that when $\ell(J) \leq \ell(I)$:

- $F(I, J) = F_K(\langle I, J \rangle, J, \langle I, J \rangle)$ when $\text{rdist}(I, J) > 3$,
- $F(I, J) = \tilde{F}_K(I, J, I) + F_W(J) + F_K(J, J, I)$ when $\text{rdist}(I, J) \leq 3$

and

$$\tilde{F}(I, J, K) = L(\ell(I)) S(\ell(J)) \sum_{j \geq 0} 2^{-j\delta} D(\text{rdist}(2^j K, \mathbb{B}))$$
Theorem ($L^2$ compactness in the special cancellation case). Let $T$ be a linear operator with a compact Calderón-Zygmund kernel satisfying the weak compactness condition and the special cancellation conditions $T(1) = T^*(1) = 0$. Then, $T$ is a compact operator on $L^2(\mathbb{R}^n)$. 
Theorem ($L^2$ compactness in the special cancellation case).
Let $T$ be a linear operator with a compact Calderón-Zygmund kernel satisfying the weak compactness condition and the special cancellation conditions $T(1) = T^*(1) = 0$. Then, $T$ is a compact operator on $L^2(\mathbb{R}^n)$.

Proof. We need to prove that $\langle P_M^\perp(T(f)), g \rangle$ tends to zero when $M$ tends to infinity, uniformly in $f$ and $g$. We have that

$$\langle P_M^\perp(T(f)), g \rangle = \sum_i \sum_{J \notin D_M} \langle f, \psi_I \rangle \langle g, \psi_J \rangle \langle T(\psi_I), \psi_J \rangle$$
Theorem ($L^2$ compactness in the special cancellation case).
Let $T$ be a linear operator with a compact Calderón-Zygmund kernel satisfying the weak compactness condition and the special cancellation conditions $T(1) = T^*(1) = 0$.
Then, $T$ is a compact operator on $L^2(\mathbb{R}^n)$.

Proof. We need to prove that $\langle P_M^\perp(T(f)), g \rangle$ tends to zero when $M$ tends to infinity, uniformly in $f$ and $g$. We have that

$$\langle P_M^\perp(T(f)), g \rangle = \sum_{I} \sum_{J \notin D_M} \langle f, \psi_I \rangle \langle g, \psi_J \rangle \langle T(\psi_I), \psi_J \rangle$$

For any $\epsilon > 0$ there is $M_0 \in \mathbb{N}$ depending on $\epsilon, K, T$ such that for any $M > M_0$, we have $2^{-\frac{\delta}{n}}M < \epsilon$, $M^{-\delta} < \epsilon$ and for every $I, J$, $\langle I, J \rangle \notin D_M$, $F(I, J) < \epsilon$. 
Theorem ($L^2$ compactness in the special cancellation case).
Let $T$ be a linear operator with a compact Calderón-Zygmund kernel satisfying the weak compactness condition and the special cancellation conditions $T(1) = T^*(1) = 0$.
Then, $T$ is a compact operator on $L^2(\mathbb{R}^n)$.

Proof. We need to prove that $\langle P_{\frac{1}{M}}(T(f)), g \rangle$ tends to zero when $M$ tends to infinity, uniformly in $f$ and $g$. We have that

$$\langle P_{\frac{1}{M}}(T(f)), g \rangle = \sum_I \sum_{J \notin D_M} \langle f, \psi_I \rangle \langle g, \psi_J \rangle \langle T(\psi_I), \psi_J \rangle$$

For any $\epsilon > 0$ there is $M_0 \in \mathbb{N}$ depending on $\epsilon, K, T$ such that for any $M > M_0$, we have $2^{-\frac{\delta}{n}M} < \epsilon$, $M^{-\delta} < \epsilon$ and for every $I, J, \langle I, J \rangle \notin D_M$, $F(I, J) < \epsilon$.

Then, we will prove that $|\langle P_{\frac{1}{2M}}(T(f)), g \rangle| \leq C\epsilon$. 
Because of the rate of decay in the bump lemma, we parametrize the sums according to eccentricities and relative distances:

$$\langle P_{2^M}(T(f)), g \rangle = \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e,m}} \langle f, \psi_I \rangle \langle g, \psi_J \rangle \langle T(\psi_I), \psi_J \rangle$$

where $J_{e,m} = \{ I \in D : \ell(I) = 2^e \ell(J), \ m \leq \text{rdist}(I, J) < m + 1 \}$. 
Because of the rate of decay in the bump lemma, we parametrize the sums according to eccentricities and relative distances:

$$\langle P_{2M}^\perp(T(f)), g \rangle = \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e,m}} \langle f, \psi_I \rangle \langle g, \psi_J \rangle \langle T(\psi_I), \psi_J \rangle$$

where $J_{e,m} = \{ l \in D : \ell(l) = 2^e \ell(J), m \leq \text{rdist}(l, J) < m + 1 \}$.

Notice that the family $\{(l, J) : l \in J_{e,m}\}$ can be also parameterized as $\{(l, J) : J \in l_{-e,m}\}$.

We also note that the cardinality of $J_{e,m}$ is comparable to $2^{-\min(e,0)} m^{n-1}$ while the cardinality of $J_{-e,m}$ is comparable to $2^{\max(e,0)} m^{n-1}$. 
We now bound as

$$\left| \langle P_{2M}^\perp (T(f)), g \rangle \right| \leq \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{l \in J_{e,m}} |\langle T(\psi_I), \psi_J \rangle| |\langle f, \psi_I \rangle| |\langle g, \psi_J \rangle|$$

By the bump lemma, we have

$$|\langle T(\psi_I), \psi_J \rangle| \leq 2^{-|e|} \left( \frac{1}{2} + \delta n \right) m - (n + \delta) F(I, J)$$

and so, we can estimate previous quantity by

$$\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|} \left( \frac{1}{2} + \delta n \right) m n + \delta \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,m}} |\langle f, \psi_I \rangle| |\langle g, \psi_J \rangle|$$

We divide the study into three cases: when $I, \langle I \cup J \rangle \notin \mathcal{D}_M$, when $I \in \mathcal{D}_M$, and when $\langle I \cup J \rangle \in \mathcal{D}_M$. 
We now bound as

\[ |\langle P_{2^M}^\perp (T(f)), g \rangle| \leq \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin D_{2^M}} \sum_{I \in J, m} |\langle T(\psi_I), \psi_J \rangle| |\langle f, \psi_I \rangle| |\langle g, \psi_J \rangle| \]

By the bump lemma, we have

\[ |\langle T(\psi_I), \psi_J \rangle| \leq 2^{-e(\frac{1}{2} + \frac{\delta}{n})} m^{-(n+\delta)} F(I, J) \]
We now bound as
\[
\left| \langle P_{\perp}^{1}(T(f)), g \rangle \right| \leq \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,m}} \left| \langle T(\psi_{I}), \psi_{J} \rangle \right| \langle f, \psi_{I} \rangle \langle g, \psi_{J} \rangle
\]

By the bump lemma, we have
\[
\left| \langle T(\psi_{I}), \psi_{J} \rangle \right| \leq 2^{-|e| \left(\frac{1}{2} + \frac{\delta}{n}\right)} m^{-n+\delta} F(I, J)
\]
and so, we can estimate previous quantity by
\[
\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e| \left(\frac{1}{2} + \frac{\delta}{n}\right)}}{m^{n+\delta}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,m}} F(I, J) \langle f, \psi_{I} \rangle \langle g, \psi_{J} \rangle
\]
We now bound as

\[ \left| \langle P_{2M}^\perp(T(f)), g \rangle \right| \leq \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,m}} \langle T(\psi_I), \psi_J \rangle \| \langle f, \psi_I \rangle \| \langle g, \psi_J \rangle \]

By the bump lemma, we have

\[ \left| \langle T(\psi_I), \psi_J \rangle \right| \leq 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} m^{-(n+\delta)} F(I, J) \]

and so, we can estimate previous quantity by

\[ \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{1}{m^{n+\delta}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,m}} F(I, J) \| \langle f, \psi_I \rangle \| \langle g, \psi_J \rangle \]

We divide the study into three cases: when \( I, \langle I \cup J \rangle \notin \mathcal{D}_M \), when \( I \in D_M \) and when \( \langle I \cup J \rangle \in \mathcal{D}_M \).
In the first case, we have that $F(I, J) < \epsilon$.
In the first case, we have that \( F(I, J) < \epsilon \) and then,

\[
\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e| (\frac{1}{2} + \frac{\delta}{n})}}{m^{n+\delta}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e, m}} F(I, J) |\langle f, \psi_I \rangle||\langle g, \psi_J \rangle| \]

In the first case, we have that $F(I, J) < \epsilon$ and then,

$$
\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)}}{m^{n+\delta}} \sum_{J \notin \mathcal{D}_2} \sum_{I \in J_{e,m}} F(I, J)|\langle f, \psi_I \rangle||\langle g, \psi_J \rangle| \\
\leq \epsilon \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)}}{m^{n+\delta}} \left( \sum_{I} \sum_{J \in J_{-e,m}} |\langle f, \psi_I \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{J} \sum_{I \in J_{e,m}} |\langle g, \psi_J \rangle|^2 \right)^{\frac{1}{2}}
$$
In the first case, we have that $F(I, J) < \epsilon$ and then,

$$\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{m^{n+\delta}}{m^{n+\delta}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e,m}} F(I, J) |\langle f, \psi_I \rangle||\langle g, \psi_J \rangle|$$

$$\leq \epsilon \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{m^{n+\delta}}{m^{n+\delta}} \left(\sum_{I} \sum_{J \in I_{-e,m}} |\langle f, \psi_I \rangle|^2\right)^{\frac{1}{2}} \left(\sum_{J} \sum_{I \in J_{e,m}} |\langle g, \psi_J \rangle|^2\right)^{\frac{1}{2}}$$

$$= \epsilon \sum_{e \in \mathbb{Z}} \sum_{n \in \mathbb{N}} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{m^{n+\delta}}{m^{n+\delta}} \left(2^{\max(e,0)} m^{n-1} \sum_{I} |\langle f, \psi_I \rangle|^2\right)^{\frac{1}{2}}$$

$$\left(2^{-\min(e,0)} m^{n-1} \sum_{J} |\langle g, \psi_J \rangle|^2\right)^{\frac{1}{2}}$$
In the first case, we have that $F(I, J) < \epsilon$ and then,

$$
\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e|(1/2 + \delta/n)}}{m^{n+\delta}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e,m}} F(I, J) \langle f, \psi_I \rangle \langle g, \psi_J \rangle \leq \epsilon \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e|(1/2 + \delta/n)}}{m^{n+\delta}} \left( \sum_{I \in I_{-e,m}} \sum_{J \in J_{e,m}} |\langle f, \psi_I \rangle|^2 \right)^{1/2} \left( \sum_{J \in J_{e,m}} \sum_{I \in I_{e,m}} |\langle g, \psi_J \rangle|^2 \right)^{1/2}
$$

$$
= \epsilon \sum_{e \in \mathbb{Z}} \sum_{n \in \mathbb{N}} 2^{-|e|(1/2 + \delta/n)} \frac{m^{n-1}}{m^{n+\delta}} \left( 2^{\max(e,0)} m^{n-1} \sum_{I} |\langle f, \psi_I \rangle|^2 \right)^{1/2} \left( 2^{-\min(e,0)} m^{n-1} \sum_{J} |\langle g, \psi_J \rangle|^2 \right)^{1/2}
$$

$$
\leq \epsilon \sum_{e \in \mathbb{Z}} 2^{-|e|(1/2 + \delta/n)} 2^{\frac{|e|}{2}} \sum_{n \in \mathbb{N}} m^{-(1+\delta)} \|f\|_2 \|g\|_2
$$
In the first case, we have that $F(I, J) < \epsilon$ and then,

$$\sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|} \frac{1}{m^{n+\delta}} \sum_{J \notin D_{2M}} \sum_{I \in J_{e, m}} F(I, J) |\langle f, \psi_I \rangle||\langle g, \psi_J \rangle|$$

$$\leq \epsilon \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} 2^{-|e|} \frac{1}{m^{n+\delta}} \left( \sum_{I} \sum_{J \in I_{-e, m}} |\langle f, \psi_I \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{J} \sum_{I \in J_{e, m}} |\langle g, \psi_J \rangle|^2 \right)^{\frac{1}{2}}$$

$$= \epsilon \sum_{e \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \frac{2^{-|e|}}{m^{n+\delta}} \left( 2^{\max(e, 0)} m^{n-1} \sum_{I} |\langle f, \psi_I \rangle|^2 \right)^{\frac{1}{2}} \left( 2^{-\min(e, 0)} m^{n-1} \sum_{J} |\langle g, \psi_J \rangle|^2 \right)^{\frac{1}{2}}$$

$$\leq \epsilon \sum_{e \in \mathbb{Z}} 2^{-|e|} \frac{1}{m^{n+\delta}} 2 \frac{|e|}{2} \sum_{n \in \mathbb{N}} m^{-(1+\delta)} \|f\|_2 \|g\|_2$$

$$= \epsilon \left( \sum_{e \in \mathbb{Z}} 2^{-|e|} \frac{\delta}{n} \sum_{m \in \mathbb{N}} m^{-(1+\delta)} \right) \|f\|_2 \|g\|_2$$
The case $I \in D_M$ implies $2^{-M} \leq \ell(I) \leq 2^M$ and $\text{rdist}(I, D_{2^M}) \leq M$. 

We separate the study into the three following cases:

1. $\ell(J) \geq 2^2^M$,
2. $\ell(J) \leq 2^{-2^M}$ and $\text{rdist}(J, B_{2^2^M}) > 2^M$.

In the first case, we note that $2^e = \ell(I) \ell(J) - 1 \leq 2^M 2^{-2^M} = 2^{-M}$ which implies $e \leq -M$.

Now, since $F(I, J) \lesssim 1$, we can bound by

$$
\sum_{e \leq -M} \sum_{m \geq 1} \frac{1}{2^{|e|}(1 + \delta n)^m} \leq \left(\sum_{e \leq -M} 2^{-M} - |e| \delta \frac{1}{n} \sum_{m \geq 1} (1 + \delta)^m\right) \|f\|_2 \|g\|_2 
\leq 2^{-M} \delta \frac{1}{n} \sum_{m \geq 1} \left(1 + \delta\right)^m \|f\|_2 \|g\|_2 < \epsilon \|f\|_2 \|g\|_2
$$


The case $I \in \mathcal{D}_M$ implies $2^{-M} \leq \ell(I) \leq 2^M$ and $\text{rdist}(I, \mathbb{D}_{2^M}) \leq M$. We separate the study into the three following cases: $\ell(J) \geq 2^{2M}$, $\ell(J) \leq 2^{-2M}$ and $\text{rdist}(J, \mathbb{B}_{2^{2M}}) > 2M$. 
The case $I \in \mathcal{D}_M$ implies $2^{-M} \leq \ell(I) \leq 2^M$ and $\text{rdist}(I, \mathcal{D}_{2^M}) \leq M$. We separate the study into the three following cases: $\ell(J) \geq 2^{2M}$, $\ell(J) \leq 2^{-2M}$ and $\text{rdist}(J, \mathcal{B}_{2^M}) > 2M$.

In the first case, we note that

$$2^e = \ell(I)\ell(J)^{-1} \leq 2^M2^{-2M} = 2^{-M}$$

which implies $e \leq -M$. 
The case \( I \in \mathcal{D}_M \) implies \( 2^{-M} \leq \ell(I) \leq 2^M \) and \( \text{rdist}(I, \mathbb{D}_{2M}) \leq M \). We separate the study into the three following cases: \( \ell(J) \geq 2^{2M} \), \( \ell(J) \leq 2^{-2M} \) and \( \text{rdist}(J, B_{2^{2M}}) > 2M \).

In the first case, we note that

\[
2^e = \ell(I)\ell(J)^{-1} \leq 2^M 2^{-2M} = 2^{-M}
\]

which implies \( e \leq -M \). Now, since \( F(I, J) \lesssim 1 \), we can bound by

\[
\sum_{e \leq -M} \sum_{m \geq 1} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{1}{m^{n+\delta}} \sum_{J \notin \mathcal{D}_{2M}} \sum_{I \in J_{e,n}} |\langle f, \psi_I \rangle| |\langle g, \psi_J \rangle| \\
\leq \sum_{e \leq -M} \sum_{m \geq 1} 2^{-|e|\left(\frac{1}{2} + \frac{\delta}{n}\right)} \frac{1}{m^{n+\delta}} 2^{-\frac{|e|}{2}} m^{n-1} \|f\|_2 \|g\|_2 \\
= \left( \sum_{e \leq -M} 2^{-|e|\frac{\delta}{n}} \sum_{m \geq 1} n^{-(1+\delta)} \right) \|f\|_2 \|g\|_2 \\
\leq 2^{-M \frac{\delta}{n}} \|f\|_2 \|g\|_2 < \epsilon \|f\|_2 \|g\|_2
\]
The case $\ell(J) < 2^{-2M}$ is symmetrical, changing $e \leq -M$ by $e \geq M$. 
The case $\ell(J) < 2^{-2M}$ is symmetrical, changing $e \leq -M$ by $e \geq M$.

Finally, in the case $\text{rdist}(J, \mathbb{B}_{2^{2M}}) \geq 2M$ we have $\ell(J) \leq 2^{2M}$ and $|c(J)| \geq 2M2^{2M}$. 
The case $\ell(J) < 2^{-2M}$ is symmetrical, changing $e \leq -M$ by $e \geq M$.

Finally, in the case $\text{rdist}(J, B_{2^M}) \geq 2M$ we have $\ell(J) \leq 2^M$ and $|c(J)| \geq 2M2^M$.

On the other hand, $I \in D_M$ implies $|c(I)| \lesssim M2^M$ and so,

$$\text{diam}(I, J) \geq |c(I) - c(J)| \geq M2^M$$

With both things,

$$m + 1 \geq \frac{\text{diam}(I, J)}{\max(\ell(I), \ell(J))} \geq 2^{-2M}M2^M > M$$
The case $\ell(J) < 2^{-2M}$ is symmetrical, changing $e \leq -M$ by $e \geq M$.

Finally, in the case $\text{rdist}(J, B_{2M}) \geq 2M$ we have $\ell(J) \leq 2^{2M}$ and $|c(J)| \geq 2M2^{2M}$.

On the other hand, $I \in D_M$ implies $|c(I)| \lesssim M2^M$ and so,

$$\text{diam}(I, J) \geq |c(I) - c(J)| \geq M2^M$$

With both things,

$$m + 1 \geq \frac{\text{diam}(I, J)}{\max(\ell(I), \ell(J))} \geq 2^{-2M}M2^{2M} > M$$

and we can bound by

$$\sum_{e \in \mathbb{Z}} \sum_{m \geq M} 2^{-|e|(1 + \frac{\delta}{n})} \sum_{J \in D_{2M}^c} \sum_{l \in J_{e, n}} |\langle f, \psi_l \rangle||\langle g, \psi_J \rangle|$$

$$\leq \left( \sum_{e \in \mathbb{Z}} 2^{-|e|\delta} \sum_{m \geq M} m^{-(1+\delta)} \right) \|f\|_2 \|g\|_2$$

$$\lesssim M^{-\delta} \|f\|_2 \|g\|_2 < \epsilon \|f\|_2 \|g\|_2$$
The general case: paraproducts

When \( T(1) = b_1, T^*(1) = b_2 \in \text{CMO}(\mathbb{R}^n) \), we construct operators \( T_{b_i} \) such that they have compact Calderón-Zymund kernels, are compact on \( L^p(\mathbb{R}^n) \) and satisfy \( T_{b_i}(1) = b_i, \ T^*_{b_i}(1) = 0 \). This way,

\[
T - T_{b_1} - T^*_{b_2}
\]

is compact on \( L^p(\mathbb{R}^n) \) and so it is the initial operator \( T \).
The general case: paraproducts

When \( T(1) = b_1, T^*(1) = b_2 \in \text{CMO}(\mathbb{R}^n) \), we construct operators \( T_{b_i} \) such that they have compact Calderón-Zymund kernels, are compact on \( L^p(\mathbb{R}^n) \) and satisfy \( T_{b_i}(1) = b_i, \, T^*_{b_i}(1) = 0 \). This way,

\[
T - T_{b_1} - T^*_{b_2}
\]

is compact on \( L^p(\mathbb{R}^n) \) and so it is the initial operator \( T \).

Lemma. Given a function \( b \) in \( \text{CMO}(\mathbb{R}^n) \), there exists a compact linear operator \( T_b \) such that \( \langle T_b(1), g \rangle = \langle b, g \rangle \) and \( \langle T_b(f), 1 \rangle = 0 \).
The general case: paraproducts

When \( T(1) = b_1, T^*(1) = b_2 \in \text{CMO}(\mathbb{R}^n) \), we construct operators \( T_{b_i} \) such that they have compact Calderón-Zymund kernels, are compact on \( L^p(\mathbb{R}^n) \) and satisfy \( T_{b_i}(1) = b_i, \ T_{b_i}^*(1) = 0 \). This way,

\[
T - T_{b_1} - T_{b_2}
\]

is compact on \( L^p(\mathbb{R}^n) \) and so it is the initial operator \( T \).

**Lemma.** Given a function \( b \) in \( \text{CMO}(\mathbb{R}^n) \), there exists a compact linear operator \( T_b \) such that

\[
\langle T_b(1), g \rangle = \langle b, g \rangle \quad \text{and} \quad \langle T_b(f), 1 \rangle = 0.
\]

**Proof.** Let \( (\psi_I)_I \) be a wavelets basis on \( L^2 \). We denote by \( \phi_I \) a bump function adapted to \( I \) such that \( \hat{\phi}_I \) is adapted to an cube of side length comparable with \( \ell(I)^{-1} \) and centered at the origin.

Then, the linear operator

\[
\langle T_b(f), g \rangle = \sum_{I \in \mathcal{D}} \langle b, \psi_I \rangle \langle f, \phi_I \rangle \langle g, \psi_I \rangle
\]

satisfies all the required properties.
For the proof of its compactness we need to check whether $P_M \circ T_b$ converges to $T_b$ in the operator norm.
For the proof of its compactness we need to check whether $P_M \circ T_b$ converges to $T_b$ in the operator norm. Since

$$\langle P^\perp_M(T_b(f)), g \rangle = \sum_{I \notin \mathcal{D}_M} \langle b, \psi_I \rangle \langle f, \phi_I \rangle \langle g, \psi_I \rangle$$

$$= \langle P^\perp_M(b), \sum_{I \in \mathcal{D}} \langle f, \phi_I \rangle \langle g, \psi_I \rangle \psi_I \rangle$$
For the proof of its compactness we need to check whether $P_M \circ T_b$ converges to $T_b$ in the operator norm. Since

$$\langle P_M^\perp(T_b(f)), g \rangle = \sum_{I \notin \mathcal{D}_M} \langle b, \psi_I \rangle \langle f, \phi_I \rangle \langle g, \psi_I \rangle \langle f, \phi_I \rangle \langle g, \psi_I \rangle \psi_I$$

$$= \langle P_M^\perp(b), \sum_{I \in \mathcal{D}} \langle f, \phi_I \rangle \langle g, \psi_I \rangle \psi_I \rangle$$

by Carleson Embedding Theorem

$$|\langle P_M^\perp(T(f)), g \rangle| \lesssim \|P_M^\perp(b)\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

which tends to zero when $M$ tends to infinity uniformly in $f$ and $g$. 
To end, we need to prove that this family of operators also belong to the class of operators for which the theory applies.
To end, we need to prove that this family of operators also belong to the class of operators for which the theory applies. From the definition

\[ \langle T_b(f), g \rangle = \sum_{I \in \mathcal{D}} \langle b, \psi_I \rangle \langle f, \phi_I \rangle \langle g, \psi_I \rangle \]

\[ = \int f(t)g(x) \sum_{I \in \mathcal{D}} \langle b, \psi_I \rangle \phi_I(t) \psi_I(x) dt dx \]
To end, we need to prove that this family of operators also belong to the class of operators for which the theory applies. From the definition

\[ \langle T_b(f), g \rangle = \sum_{l \in \mathcal{D}} \langle b, \psi_l \rangle \langle f, \phi_l \rangle \langle g, \psi_l \rangle \]

\[ = \int f(t)g(x) \sum_{l \in \mathcal{D}} \langle b, \psi_l \rangle \phi_l(t)\psi_l(x) dt dx \]

we write the operator kernel and check the properties of a compact Calderón-Zygmung kernel:

\[ |\partial_t K(t, x)| \lesssim \| P_M^\perp(b) \|_{\text{BMO}(\mathbb{R}^n)} \frac{1}{|t - x|^2} \]
Thank you for your attention
Lemma. The smooth condition

\[ |K(t,x) - K(t',x')| \lesssim \frac{(|x - x'| + |t - t'|)\delta}{|t - x|^{n+\delta}} \]

for \(2(|x - x'| + |t - t'|) \leq |t - x|\) and \(\lim_{|t-x|\to\infty} K(t,x) = 0\) imply the decay condition

\[ |K(t,x)| \lesssim \frac{1}{|t - x|^n} \]
Lemma. The smooth condition
\[ |K(t, x) - K(t', x')| \lesssim \frac{(|x - x'| + |t - t'|)^\delta}{|t - x|^{n+\delta}} \]
for \(2(|x - x'| + |t - t'|) \leq |t - x|\) and \(\lim_{|t-x|\to\infty} K(t, x) = 0\) imply the decay condition
\[ |K(t, x)| \lesssim \frac{1}{|t - x|^n} \]

Proof.
\[ |K(t, x)| \leq |K(t', x')| + C \frac{(|x - x'| + |t - t'|)^\delta}{|t - x|^{n+\delta}} \]
\[ \leq |K(t', x')| + C2^{-\delta} \frac{1}{|t - x|^n} \]
**Lemma.** The smooth condition

$$|K(t, x) - K(t', x')| \lesssim \frac{(|x - x'| + |t - t'|)\delta}{|t - x|^{n+\delta}}$$

for $2(|x - x'| + |t - t'|) \leq |t - x|$ and $\lim_{|t - x| \to \infty} K(t, x) = 0$ imply the decay condition

$$|K(t, x)| \lesssim \frac{1}{|t - x|^n}$$

**Proof.**

$$|K(t, x)| \leq |K(t', x')| + C\frac{(|x - x'| + |t - t'|)\delta}{|t - x|^{n+\delta}}$$

$$\leq |K(t', x')| + C2^{-\delta} \frac{1}{|t - x|^n}$$

We fix $(t, x) \in \mathbb{R}^2 \setminus \{x = y\}$ with $x > y$ and consider the sequence $(x_m, t_m)$ defined by $y_0 = x$, $t_0 = t$ and for all $m \geq 1$,

$$x_m = x_{m-1} + 1/4|x_{m-1} - t_{m-1}|$$

$$t_m = t_{m-1} - 1/4|x_{m-1} - t_{m-1}|$$
This way \((x_m, t_m) \in B_{x_{m-1}, t_{m-1}}\) and so, by previous calculation,

\[
|K(t, x)| \leq |K(x_1, t_1)| + \frac{C}{|x_0 - t_0|^n}
\]

\[
\leq |K(x_m, t_m)| + C \sum_{k=0}^{m} \frac{1}{|x_k - t_k|^n}
\]

Moreover

\[
|x_m - t_m| = \frac{3}{2} |x_{m-1} - t_{m-1}|
\]

This shows that

\[
|K(t, x)| \leq \lim_{m \to \infty} |K(x_m, t_m)| + C \sum_{k=0}^{m} \frac{1}{|x_k - t_k|^n} = C |t - x|^n
\]
This way \((x_m, t_m) \in B_{x_{m-1}, t_{m-1}}\) and so, by previous calculation,

\[
|K(t, x)| \leq |K(x_1, t_1)| + \frac{C}{|x_0 - t_0|^n}
\]

\[
\leq |K(x_m, t_m)| + C \sum_{k=0}^{m} \frac{1}{|x_k - t_k|^n}
\]

Moreover

\[
|x_m - t_m| = |x_{m-1} - t_{m-1} + 1/2|x_{m-1} - t_{m-1}||
\]

\[
= 3/2|x_{m-1} - t_{m-1}| = (3/2)^m|x_0 - t_0| = (3/2)^m|t - x|
\]
This way \((x_m, t_m) \in B_{x_{m-1}, t_{m-1}}\) and so, by previous calculation,

\[
|K(t, x)| \leq |K(x_1, t_1)| + \frac{C}{|x_0 - t_0|^n}
\]

\[
\leq |K(x_m, t_m)| + C \sum_{k=0}^{m} \frac{1}{|x_k - t_k|^n}
\]

Moreover

\[
|x_m - t_m| = |x_{m-1} - t_{m-1} + 1/2|x_{m-1} - t_{m-1}||
\]

\[
= 3/2||x_{m-1} - t_{m-1}| = (3/2)^m|x_0 - t_0| = (3/2)^m|t - x|
\]

This shows that

\[
|K(t, x)| \leq \lim_{m \to \infty} |K(x_m, t_m)| + C \sum_{k=0}^{\infty} \frac{1}{(3/2)^{kn}|t - x|^n} = \frac{C}{|t - x|^n}
\]
Lemma. Let $K$ satisfying the smoothness condition and such that the operator $T$ associated with $K$ is weakly bounded. Then, $K$ satisfies the decay condition.

Proof. Let $\Phi \in S(\mathbb{R}^n)$ supported and $L^\infty$-adapted to $B$ with $\int \Phi(x) \, dx = 1$. For every $t, x \in \mathbb{R}^n$, $t > x$ and $\lambda = \frac{|t - x|}{2}$, we define the functions $f = T_t D_1 \lambda \Phi$ and $g = T_x D_1 \lambda \Phi$. Since $\lambda^{n/2} f$ and $\lambda^{n/2} g$ are both $L^2$-adapted to $\langle t, x \rangle$ with the same constant and order, by the weak boundedness condition $|\langle T(f), g \rangle| \leq C \lambda^{-n} = C |t - x|^{-n}$.

On the other hand, since $2(|u| + |y|) < 2\lambda = |t - x|$ and $K$ satisfies the smoothness condition, $|\langle T(f), g \rangle - K(t, x)\rangle| \leq \left| \int \int D_1 \lambda \Phi(u) D_1 \lambda \Phi(y) (K(u + t, y + x) - K(t, x)) \, dudy \right| \leq C \lambda \delta |t - x|^{-n} + \delta \leq C \lambda \delta |t - x|^{-n}$.
Lemma. Let $K$ satisfying the smoothness condition and such that the operator $T$ associated with $K$ is weakly bounded. Then, $K$ satisfies the decay condition.

Proof. Let $\Phi \in S(\mathbb{R}^n)$ supported and $L^\infty$-adapted to $\mathcal{B}$ with $\int \Phi(x)dx = 1$. 

\[
\int_{\mathbb{R}^n} \Phi(x)dx = 1.
\]
Lemma. Let $K$ satisfying the smoothness condition and such that the operator $T$ associated with $K$ is weakly bounded. Then, $K$ satisfies the decay condition.

Proof. Let $\Phi \in S(\mathbb{R}^n)$ supported and $L^\infty$-adapted to $\mathcal{B}$ with $\int \Phi(x)dx = 1$.

For every $t, x \in \mathbb{R}^n$, $t > x$ and $\lambda = |t - x|/2$, we define the functions $f = \mathcal{T}_t \mathcal{D}^{1}_\lambda \Phi$ and $g = \mathcal{T}_x \mathcal{D}^{1}_\lambda \Phi$.

Since $\lambda^{n/2}f$ and $\lambda^{n/2}g$ are both $L^2$-adapted to $\langle t, x \rangle$ with the same constant and order, by the weak boundedness condition

$$|\langle T(f), g \rangle| \leq C\lambda^{-n} = C|t - x|^{-n}$$
Lemma. Let $K$ satisfying the smoothness condition and such that the operator $T$ associated with $K$ is weakly bounded. Then, $K$ satisfies the decay condition.

Proof. Let $\Phi \in S(\mathbb{R}^n)$ supported and $L^\infty$-adapted to $B$ with 
\[
\int \Phi(x)dx = 1.
\]
For every $t, x \in \mathbb{R}^n$, $t > x$ and $\lambda = |t - x|/2$, we define the functions 
\[
f = T_t D_{\lambda}^1 \Phi \quad \text{and} \quad g = T_x D_{\lambda}^1 \Phi.
\]
Since $\lambda^{n/2} f$ and $\lambda^{n/2} g$ are both $L^2$-adapted to $\langle t, x \rangle$ with the same constant and order, by the weak boundedness condition
\[
|\langle T(f), g \rangle| \leq C \lambda^{-n} = C |t - x|^{-n}
\]
On the other hand, since $2(|u| + |y|) < 2\lambda = |t - x|$ and $K$ satisfies the smoothness condition,
\[
|\langle T(f), g \rangle - K(t, x)\rangle| \leq \left| \int \int D_{\lambda}^1 \Phi(u) D_{\lambda}^1 \Phi(y) (K(u+ t, y+ x) - K(t, x)) dudy \right|
\leq C \int \int D_{\lambda}^1 \Phi(u) D_{\lambda}^1 \Phi(y) \frac{(|u| + |y|)^\delta}{|t - x|^{n+\delta}} dudy \leq C \lambda^\delta |t - x|^{-(n+\delta)} \leq C |t - x|^{-n}
\]