Signal Analysis in Relation to Hardy Spaces

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Part I. Introduction

”... the interpretation of IF is often a subject of controversy”. This is to be compared what is said in the same paper: ”Clearly, the concept of frequency is unambiguous”.

\[ IF = \text{instantaneous frequency} \]

This situation has lasted until now.
A closely related concept is monocomponent. In the same literature a signal is called a monocomponent if, for this signal, there is only one frequency or a narrow range of frequencies varying as a function of time; and, it is called a multicomponent if it is not a monocomponent. Before the frequency concept is practically and theoretically defined, such definition of monocomponent signals is logically invalid. Vague definitions of IF and monocomponent are useless for signal analysis.
It is commonly accepted that for a real-valued signal $s(t)$ its instantaneous frequency is the analytic phase derivative $\theta'(t)$ defined through the related analytic signal

$$s(t) + iHs(t) = \rho(t)e^{i\theta(t)},$$

where $H$ stands the Hilbert transformation in the context, and

$$\text{IF}_s(t) = \theta'(t).$$

This idea is due to Gabor in 1946.
This definition is, again, invalid due to two reasons.

(i) If we talk about arbitrary signal of finite energy, then the analytic phase derivative $\theta'(t)$ in the classical sense may not exist; and

(ii) If the phase derivative exists, it may not satisfy the requirement $\theta'(t) \geq 0$, say, for almost all $t$ in its domain of definition.

The non-negativity property (ii) is desired not only because of the physics sense of frequency but also because of the need of frequency analysis.
Part I. Introduction: Section 1.1. Analytic Signals

It is well known

\[ s(t) + iHs(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\xi t} \hat{s}(\xi) d\xi. \]

The simplest case is \( s(t) = \cos nt \) for which

\[ s(t) + iHs(t) = e^{int}, \quad \theta(t) = nt, \quad \theta'(t) = n. \]

So, one may hope that by writing the above in a single amplitude-phase representation, as

\[ s(t) + iHs(t) = \rho(t)e^{i\theta(t)}, \]

there should hold

\[ \theta'(t) \geq 0, \quad \text{a.e.} \]

But it is not the case. Outer functions in complex analysis all fail to enjoy this property.
Gabor’s analytic signal idea, as a matter of fact, can be further carried on. It is supported by two facts. (i) 
\[
\int_{-\infty}^{\infty} \theta'(t)|s(t)|^2 dt = \int_{-\infty}^{\infty} \xi|\hat{s}(\xi)|^2 d\xi,
\]
where \( s(t) = \rho(t)e^{i\theta(t)} \), \( \rho(t) \geq 0 \); and
(ii) An ideal time-frequency distribution $P(\omega, t)$ is desired to satisfy $P(\omega, t) \geq 0$, and the conditional expectation of the frequencies at the time moment $t$ is

$$\langle \omega \rangle_t \triangleq \frac{1}{P_T(t)} \int_{-\infty}^{\infty} \omega P(\omega, t) d\omega = \theta'(t) \geq 0,$$

where

$$P_T(t) \triangleq \int_{-\infty}^{\infty} P(\omega, t) d\omega.$$

This is valid for the Dirac type time-frequency distribution of mono-component signals: $P(\omega, t) = \rho(t)\delta(\omega - \theta'(t))$. 
Part I. Section 1.2: Mono-component Function Theory
Part I. Section 1.2: Mono-component Function Theory

We take the unit disc case as sample to explain the theory. In the other contexts there are more or less parallel theories. Let $D$ denote the unit disc in the complex plane $\mathbb{C}$. We will be using the complex Hardy spaces defined as follows.

$$H^p(D) = \{ f : D \rightarrow \mathbb{C} \mid f \text{ is holomorphic, and} \hspace{1cm} \|f\|_p = \sup_{0 \leq r < 1} \left( \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}} < \infty \},$$

with $0 < p < \infty$;

and

$$H^\infty(D) = \{ f : D \rightarrow \mathbb{C} \mid f \text{ is holomorphic, and} \hspace{1cm} \|f\|_\infty = \sup_{z \in D} |f(z)| < \infty \}.$$
To work on analytic signals one has to encounter Hardy spaces. The reason is that for a complex-valued signal it is of the form $s + iHs$ if and only if it is the boundary limit of a function in the Hardy spaces. We will mainly concern the $p = 2$ and $p = \infty$ cases. Hardy spaces are fundamental in the study of signal analysis.
**Definition** Let $f$ be a function in a Hardy space. If in the amplitude-phase representation of $f$, 
$$f(z) = \rho_r(t)e^{i\theta_r(t)}, \quad z = re^{it},$$
the limits
$$\lim_{r \to 1^-} \theta'_r(t) \quad (\ast)$$
exist for almost all $t \in [0, 2\pi)$, and, by denoting the limit function as $\theta'(t)$, there hold
$$\theta'(t) \geq 0, \quad \text{a.e.}, \quad (\ast\ast)$$
then $f$ is said to be a **mono-component (MC)**. If $f = s + iHs$ is a mono-component, where $s$ is real-valued, then $s$ is also called a mono-component. If $f$ is a mono-component, then the corresponding function $\theta'(t)$ is defined to be the **instantaneous frequency (IF)** of $f(t)$, as well as that of $s(t)$. 
The above definition amounts to say that signals with well defined IFs are just mono-components. As already mentioned, outer functions are analytic signals but not mono-components. Therefore, the set of mono-components is a proper subset of that for analytic signals.

The conditions (*) and (**) rule out many signals. MCs, in fact, only form a very restrictive set. A MC is essentially the boundary limit of a good conformal mapping of the unit disc. For signals defined in the whole time range there is a counterpart theory.
The simplest MCs are trigonometric functions $e^{int}, n = 0, 1, 2, ...$. MCs are closed under multiplication but not under addition. MCs do not form a linear space.
The primary task now is two-fold.

(i) Find a pool of MCs containing a large variety of different kinds of MCs.

(ii) Seek for appropriate MC-decompositions of signals for the signal analysis purposes.
The basic types of MCs include the following.

(i) We call signals of the form $f(t) = e^{i\theta(t)}$ phase signals. It is proved that a phase signal is a MC if and only if it is the boundary limit of an inner function (in the Nevanlinna sense). Inner functions include Blaschke products (with finitely and infinitely many zeros) and singular inner functions generated by measures singular to the Lebesgue measure.
The simplest and non-trivial example is the boundary limit of a Möbius transform. Since

\[ \tau_a(z) = \frac{z - a}{1 - \overline{a}z} \]

maps the unit circle to the unit circle, letting

\[ \theta_a(t) = \frac{1}{i} \log \frac{e^{it} - a}{1 - \overline{a}e^{it}}, \]

we have

\[ \tau_a(e^{it}) = e^{i\theta_a(t)}, \text{ and } \theta'_a(t) = \frac{1 - |a|^2}{|e^{it} - a|^2} = 2\pi P_a(t) > 0. \]
Therefore, Möbius transforms and finite Blaschke products all belong to the MC class. It took a few years to prove the same result for infinite Blaschke products, and, more generally, for inner functions. We proved (Tao Qian, 2009)

**Theorem** If $e^{i\theta(t)}$ is the non-tangential boundary limit of an inner function, then $\theta'(t) \triangleq \lim_{r \to 1^-} \theta'_r(t) > 0$, a.e.; If $
abla(t)e^{i\theta(t)}$ is the non-tangential boundary limit of an outer function, then under mild conditions of the Sobolev type $\theta'(t)$ a.e. exists and

$$\int_{0}^{2\pi} \theta'(t)dt = 0.$$
The first half of the above theorem is essentially the Julia-Wolff-Carathéodory Theorem (1030’s)

**Theorem** Let $f$ be analytic, $\zeta : \mathbb{D} \to \mathbb{D}$, and $\sigma, \zeta \in \partial \mathbb{D}$. Then

$$
\lim_{S : z \to \zeta} \frac{\sigma - f(z)}{\zeta - z} = \sigma \bar{\zeta} \beta_f(\zeta, \sigma),
$$

where

$$
\beta_f(\zeta, \sigma) = \sup_{z \in \mathbb{D}} \left[ \frac{|\sigma - f(z)|^2}{1 - |f(z)|^2} / \frac{|\zeta - z|^2}{1 - |z|^2} \right].
$$

If $\beta_f(\zeta, \sigma)$ is finite, then

$$
\lim_{S : z \to \zeta} f(z) = \sigma, \text{ and } \lim_{S : z \to \zeta} f'(z) = \sigma \bar{\zeta} \beta_f(\zeta, \sigma).
$$
The above proved theorem is a foundation of digital signal processing (DSP) in relation to the subjects all-pass filters, minimum phase signals and energy delay, etc. It seems to be the first time that those results are rigorously proved although they have appeared in the DPS literature for some time.
(ii) There is a class of MCs of the form \( f(t) = \rho(t)e^{i\theta(t)} \) in which \( e^{i\theta(t)} \) itself is a MC (So, according to the above theorem it is the non-tangential boundary limit of n inner function) and \( \rho \) is real-valued. The study of such type of MCs arose a new phase of study of the Bedrosian identity. In such class the signals satisfy the relation \( H(\rho e^{i\theta}) = \rho H(e^{i\theta}) = -i \rho e^{i\theta} \). Signals in such class are called the **Bedrosian-type MCs**.
The following theorem is one of the best results in this direction (Lihui Tan and Tao Qian, preprint)

**Theorem** Let $e^{i\theta(t)}$ be the non-tangential boundary limit of a finite or infinite Blaschke product, and $\rho(t)$ is real-valued. Then $\rho e^{i\theta}$ belongs to $H^p(D)$ if and only if $\rho$ belongs to $\overline{\text{span}}^p \{B_k\}_{k=1}^\infty$, where $\{B_k\}_{k=1}^\infty$ is the orthogonal rational system generated by the zeros, including the multiples, of the Blaschke product.

We note that the closure in $L^p$, $\overline{\text{span}}^p \{B_k\}_{k=1}^\infty$, is the backward shift invariant subspace generated by the zeros of the Blaschke product.
(iii) The complement set of the Bedrosian-type signals in the whole MCs consists of those of the form $\rho(t)e^{i\theta(t)}$ in which the phase signal part is not a MC. The non-Bedrosian-type MCs include starlike and $p$-starlike signals, etc.

In a recent paper of Lihui Tan and Tao Qian close relationships between non-Bedrosian-type mono-components and $p$-starlike functions are explored.
Weighted Blaschke Products

\[ B_n(z) = B_{b_1, ..., b_n}(z) = e_{b_n}(z) \prod_{l=1}^{n-1} \frac{z - b_l}{1 - \overline{b}_l z}, \]

where \( b_1, ..., b_n \) are arbitrary complex numbers in the unit disc, \( e_{b_n} \) is the normalized Szegö kernel given by

\[ e_b(z) = \frac{\sqrt{1 - |b|^2}}{1 - \overline{b}z}. \]

A weighted Blaschke product is theoretically a PMC (After being multiplied by \( e^{it} \) it becomes a MC), but often an MC. In particular, if one of the \( b_k \)'s is zero, then \( B_n \) is a MC.
Next we seek for appropriate MC decompositions of signals. Let us consider all possible representations of a signal $f$ in $H^2$ into a sum of a finite or infinite number of MCs. Denote by $MC$ the set of MCs. Let

$$n_f = \min \{ n \mid f = \sum_{k=1}^{n} m_k, m_k \in MC, k = 1, \ldots, n \}.$$
The idea is that expressions in linear expansions of smaller numbers of MCs would be more stable, and the extremal case $n_f = 1$ gives the uniqueness. On the other hand, since $MC$ is not linearly independent, for $n_f > 1$, there may be more than one different decompositions of $f$ with exactly $n_f$ linearly independent terms. This shows that even for the smallest number of expanding terms we may not have the uniqueness.
Part I. Section 1.2: Mono-component Function Theory

The other observation is that given any $f \in H^2$ and $\epsilon > 0$ there exist two MCs $m_1, m_2$ such that

$$\| f - (m_1 + m_2) \|_2 \leq \epsilon.$$  

This exhibits that, with the tolerance $\epsilon$, in the self-explanatory notation, we have $n_f(\epsilon) = 2$. These two observations show that one should restrict oneself to a linearly independent subclass of $MC$ to gain stability or uniqueness.
Part I. Section 1.2: Mono-component Function Theory

Let $\mathcal{B}$ be a linear independent subset of $\mathcal{MC}$. Denote

$$n_f(\mathcal{B}) = \min\{n \mid f = \sum_{k=1}^{n} m_k, m_k \in \mathcal{B}, k = 1, \ldots, n\}.$$ 

Then, if $n_f(\mathcal{B}) < \infty$, then there exists the unique expansion of $f$:

$$f = \sum_{k=1}^{n_f(\mathcal{B})} m_k, \quad m_k \in \mathcal{B}, k = 1, 2, \ldots, n_f(\mathcal{B}).$$

If $n_f(\mathcal{B}) = \infty$, or one does not know whether $n_f(\mathcal{B})$ is finite or not, then one would have different strategies to get stability of decomposition of $f$. 
Part I. Section 1.2: Mono-component Function Theory

The reason to seek for the unique or a stable decomposition is to define a canonical decomposition of a signal. This would enable us to study signals and do signal processing. Such decomposition should be dependent on individual signal and thus be adaptive.
Part I. Section 1.2: Mono-component Function Theory

Strategies: (i) In $\mathcal{B}$ select $m_1, m_2, \ldots$, one by one such that whenever $m_1, \ldots, m_k$ already selected, we select $m_{K+1}$ such that

$$\|f - (m_1 + \cdots + m_k + m_{k+1})\|_2$$

has the minimum energy of all possible. This strategy merges to the principle of greedy algorithm.

(ii) For a given $\epsilon > 0$, find the minimum $n$ and the corresponding MCs in $\mathcal{B}$, namely $m_1, \ldots, m_n$, such that

$$\|f - (m_1 + \cdots + m_n)\|_2 \leq \epsilon.$$

(iii) Given $n$, find $n$ MCs in $\mathcal{B}$ such that

$$\|f - (m_1 + \cdots + m_n)\|_2$$

reaches the minimum of all possible.
Part I. Section 1.3: Adaptive Fourier Decomposition
We proceed with a type of mono-component decompositions that does not use a subset of CM but a dictionary consisting of MMCs. It, however, ends up with an adaptive mono-component expansion. It is of the Fourier type. It is not a greedy algorithm.
Part I. Section 1.3: Adaptive Fourier Decomposition (AFD)

By the Fourier type decompositions we mean decompositions in relation to orthonormal rational systems

\[ \{B_k\}_{k=1}^n, \]

where

\[ B_k(z) = \sqrt{1 - |a_k|^2} \prod_{i=1}^{k-1} \frac{z - a_i}{1 - \overline{a_k} z}, \quad z \in \mathbb{D}, \text{ the unit disc}, \]

where the parameter \( a_k \)'s all belong to the open unit disc. These systems have several names: Takenaka-Malmquist (TM) systems, orthogonal rational systems, etc. Besides the unit circle, in other contexts, such as the real-line, the Euclidean space, the \( n \)-torus, and the \( n \)-spheres, etc., there are counterpart systems.
If $a_i = 0$ for all $i = 1, \ldots, k$, then $B_k(z) = z^{k-1}$, and the corresponding TM system becomes a partial Fourier system, and the corresponding decomposition Fourier series.
For general \( a_i \in \mathbb{D} \), the system consists of rational functions. In fact, it is the \textbf{G-S orthogonalization process} applied to the \textbf{partial rational} functions

\[
\frac{1}{1 - a_1 z}, \ldots, \frac{1}{(1 - a_1 z)^{m_1}}, \ldots, \frac{1}{1 - a_n z}, \ldots, \frac{1}{(1 - a_n z)^{m_n}},
\]

where \( a_i \) repeats \( m_i \) times, \( i = 1, \ldots, n \). If \( a_i \) is zero, say, \( i = 1 \), then the first \( m_1 \) terms in the above list is replaced by

\[1, z, \ldots, z^{m_1-1}.
\]

Due to the relation between rational functions and the partial fractions we learn that TM systems cannot be avoided.
Studies of TM system and rational approximation have a long history and are with many significant applications in both pure and applied mathematics. In all the traditional studies the condition (hyperbolic non-separable)

$$\sum_{k=1}^{\infty} (1 - |a_k|) = \infty$$

is assumed under which, and only under which, the corresponding TM system is a basis in $H^p, 1 \leq p \leq \infty$. 
Our studies have emphasis on adaptive selection of the parameters $a_k$’s. This practice may violate the usual expectation of being a basis. AFD offers a fast expansion of a given signal by using suitable or most suitable basic functions in the “dictionary” consisting of Szegö kernels. This merges to the idea of greedy algorithm, compressed sensing, ... All those, including AFD, belong to the notion of sparse representation of signals.
The philosophy is to add one more dimension. Let $s \in L^2$ and be real-valued. The decomposition $s = s^+ + s^-$, $s^\pm = (1/2)(s \pm iHs)$ is, in fact, to bring one more dimension. That is because $b^\pm$ are boundary limits of good holomorphic functions in the interior and exterior of the unit disc. We have

$$s = \text{Re} s^+ - c_0,$$

where $c_0$ is the average of $s$ over the unit circle. In the following we also denote $s^+ = f^+$. 
AFD Formulation: Let $f^+ \in H^2(D)$, $D$ being the unit disc, $f^+ = f_1$. For any $a_1 \in D$, there holds the identity

$$f_1(z) = \langle f_1, e_{a_1} \rangle e_{a_1}(z) + \frac{f_1(z) - \langle f_1, e_{a_1} \rangle e_{a_1}(z)}{\frac{z-a_1}{1-\overline{a_1}z}} \frac{z-a_1}{1-\overline{a_1}z},$$

where $e_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$ satisfying $\|e_a\| = 1$. We particularly choose $a_1 \in D$ such that

$$a_1 = \arg \max |\langle f_1, e_a \rangle| = \arg \max |\sqrt{1-|a|^2}f_1(a)|.$$

In the Hardy space such $a_1$ is available in the interior of the disc.
Fix such an $a_1$. Let

$$f_2(z) = \frac{f_1(z) - \langle f_1, e_{a_1} \rangle e_{a_1}(z)}{z-a_1 \over 1-a_1 z}.$$ 

One does the same to $f_2$ and further has

$$f_1(z) = \langle f_1, e_{a_1} \rangle e_{a_1}(z) + \langle f_2, e_{a_2} \rangle e_{a_2}(z) \frac{z-a_1}{1-a_1 z} +$$

$$+ \frac{f_2(z) - \langle f_2, e_{a_2} \rangle e_{a_2}(z)}{z-a_2 \over 1-a_2 z} \frac{z-a_2}{1-a_2 z} \frac{z-a_1}{1-a_1 z},$$

where

$$a_2 = \arg \max |\langle f_2, e_a \rangle| = \arg \max |\sqrt{1-|a|^2} f_2(a)|.$$
Part I. Section 1.3: Adaptive Fourier Decomposition (AFD)

Let

\[ f_3(z) = \frac{f_2(z) - \langle f_2, e_{a_2} \rangle e_{a_2}(z)}{z-a_2} \frac{z-a_2}{1-a_2z} \]

and

\[ B_1(z) = e_{a_1}(z), \quad B_2(z) = e_{a_2}(z) \frac{z-a_1}{1-a_1z}. \]

We have

\[ f^+(z) = \langle f_1, e_{a_1} \rangle B_1(z) + \langle f_2, e_{a_2} \rangle B_2(z) + f_3(z) \frac{z-a_2}{1-a_2z} \frac{z-a_1}{1-a_1z}. \]
Continue this procedure with

\[ f_{k+1}(z) = \frac{f_k(z) - \langle f_k, e_{a_k} \rangle e_{a_k}(z)}{z - a_k} \frac{1}{1 - \overline{a_k}z}, \]  

(1)

and

\[ a_k = \arg \max |\langle f_k, e_a \rangle| = \arg \max |\sqrt{1 - |a|^2}f_k(a)|, \ a_k \in D. \]

We call the operator sending \( f_k \) to \( f_{k+1} \) within the Hardy space the **generalized backward shift operator induced by** \( a_k \). The availability of \( a_k \in \mathbb{D} \) is the **maximal selection principle**.
Inductively, we have

\[ f^+(z) = \sum_{l=1}^{k} \langle f_l, e_{a_l} \rangle B_l(z) + f_{k+1}(z) \prod_{l=1}^{k} \frac{z - a_l}{1 - \overline{a_l}z}, \]  \hspace{1cm} (2)

where

\[ B_l(z) = e_{a_l}(z) \prod_{l=1}^{l-1} \frac{z - a_l}{1 - \overline{a_l}z}. \]
The $a_l$ selected according to the maximal selection principle may not satisfy the non-separability condition (??). One has to show

$$f^+(z) = \sum_{l=1}^{\infty} \langle f_l, e_{a_l} \rangle B_l(z)$$

(3)

in the $L^2$ convergence sense. This is equivalent to showing

$$\lim_{k \to \infty} \| f_{k+1}(\cdot) \prod_{l=1}^{k} \frac{\cdot - a_l}{1 - a_l(\cdot)} \|^2 = \lim_{k \to \infty} \| f_{k+1} \|^2 = 0.$$

See Qian and Wang 2011
If $a_1 = 0$, then

$$\theta'_l > 0,$$ for all $l$.

In the case (??) reduces to the orthogonal expansion

$$f(e^{it}) = c_0 + \sum_{l=1}^{\infty} \rho_l(t) \cos \theta_l(t), \quad (4)$$

where $\rho_l(t) \geq 0$ and $\theta'_l(t) > 0$, and $\rho_l$ and $\theta_l$ are defined through (??) for the parameter sequence $0, a_2, ..., a_n$. The expansion (4) is a mono-component expansion (see the next section).
We at the same time obtain the Hilbert transform of \( f \):

\[
Hf(e^{it}) = \sum_{l=1}^{\infty} \rho_l(t) \sin \theta_l(t),
\]

as well as the Dirac type time-frequency distribution

\[
P(\omega, t) = \sum_{l=1}^{\infty} P_l(\omega, t) = \sum_{l=1}^{\infty} \rho_l(t) \delta(\omega - \theta'_l(t)).
\]
Part I. Section 1.4: Higher Dimensional Generalizations
Part I. Section 1.4: Higher Dimensional Generalizations
Part of the 1-D AFD and related theory have been extended to the quaternionic, Clifford algebraic, and several complex variables settings.
Part II. Fourier Spectrum Characterizations of Hardy Spaces

There are usually two subjects: Integral representation and the Fourier spectrum characterization. They are related to each other: If $f \in H^p$, then

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{it\omega} e^{-y\omega} \hat{f}(\omega) d\omega.$$
Part II. Section 2.1: On the Real Line: $1 \leq p \leq \infty$

$f \in H^p(\mathbb{R})$ if and only if $\hat{f} = \chi_+ \hat{f}$, $1 \leq p \leq \infty$.

$p = 2$: Paley-Wiener Theorem

$1 \leq p \leq \infty$: 2009 PAMS, T. Qian, B. Yu, D-Y. Yan, L-X. Yan, Y-S. Xu,

where $2 < p \leq \infty$: distribution sense

and $p = 1$ and $p = \infty$ use C-Z decomposition.
Theorem (Integral Representation Formula For Index Range $1 \leq p \leq 2$) Suppose $1 \leq p \leq 2$, $f \in L^p(\mathbb{R})$. Then $f \in H^p_+(\mathbb{R})$ if and only if supp $\hat{f} \subset [0, +\infty)$. If the condition is satisfied, then

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t)e^{itz}dt \in H^p(\mathbb{C}_+),$$

(5)

$f(x)$ are the non-tangential boundary values of function $f(z)$. 

G-T. Deng and T. Qian 2013
Theorem 1  (Hardy Spaces Decomposition of $L^p$ Functions For Index range $0 < p < 1$) Suppose that $0 < p < 1$ and $f \in L^p(\mathbb{R})$. Then, there exist a positive constant $A_p$ and two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_+)$, $Q_k \in H^p(\mathbb{C}_-)$ and

$$\sum_{k=1}^{\infty} \left( \|P_k\|_{H^p_+}^p + \|Q_k\|_{H^p_-}^p \right) \leq A_p \|f\|_p^p, \quad (1)$$

$$\lim_{n \to \infty} \|f - \sum_{k=1}^{n} (P_k + Q_k)\|_p = 0. \quad (2)$$
Moreover,

\[ g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^p(\mathbb{C}_+), \quad h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^p(\mathbb{C}_-), \]

and \( g(x) \) and \( h(x) \) are the non-tangential boundary values of functions for \( g \in H^p(\mathbb{C}_+) \) and \( h \in H^p(\mathbb{C}_-) \), respectively, \( f(x) = g(x) + h(x) \) almost everywhere, and

\[ \|f\|_p \leq \|g\|_p + \|h\|_p \leq A_p \|f\|_p, \]

that is, in the sense of \( L^p(\mathbb{R}) \),

\[ L^p(\mathbb{R}) = H^p_+(\mathbb{R}) + H^p_- (\mathbb{R}). \]
**Theorem 2** (Integral Representation Formula For Index Range $0 < p \leq 1$) If $0 < p \leq 1$, $f \in H^p(\mathbb{C}_+)$, then there exist a positive constant $A_p$, depending only on $p$, and a slowly increasing continuous function $F$ whose support is contained in $[0, \infty)$, satisfying that, for $\varphi$ in the Schwarz class $\mathcal{S}$,

\[(F, \varphi) = \lim_{y > 0, y \to 0} \int_{\mathbb{R}} f(x + iy) \hat{\varphi}(x) dx,
\]

and that

\[|F(t)| \leq A_p \|f\|_{H^p_+} |t|^{\frac{1}{p} - 1}, \quad (t \in \mathbb{R}) \tag{4}
\]

and

\[f(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} F(t) e^{itz} dt \quad (z \in \mathbb{C}_+).
\]
Theorem 3 (Fourier Spectrum Characterization for Hardy Spaces For Index Range $0 < p \leq 1$)

Let $0 < p \leq 1$, $f \in L^p(\mathbb{R})$. Then $f \in H^p_+(\mathbb{R})$ if and only if there exists a sequence of functions $\{f_n\}$ satisfying $f_n \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, supp $\hat{f}_n \subset [0, +\infty)$, and

$$\lim_{n \to \infty} \|f - f_n\|_p = 0,$$

i.e., $H^p_+(\mathbb{R})$ is the $L^p(\mathbb{R})$–closure of $L^p(\mathbb{R}) \cap H^2_+(\mathbb{R})$. 
Let $B$ be an open subset of $\mathbb{R}^n$. The tube $T_B$ can be represented as

$$T_B = \{z = x + iy \in \mathbb{C}^n : x \in \mathbb{R}^n, y \in B\}.$$

A function $F(z)$ holomorphic in the tube $T_B$ is said to belong to the space $H^p(T_B)$, $0 < p < \infty$, if

$$\|F\|_{H^p} = \sup \left\{ \left( \int_{\mathbb{R}^n} |F(x + iy)|^p dx \right)^{\frac{1}{p}} : y \in B \right\} < \infty,$$

the number $\|F\|_{H^p}$ is defined to be the $H^p$ norm of function $F(z)$.

When $p = \infty$, the Hardy space $H^\infty(T_B)$ is defined

$$H^\infty(T_B) = \{F : F \text{ holomorphic on } T_B \text{ and } \|F\|_{H^\infty} = \sup_{z \in T_B} |F(z)| < \infty\}.$$
For any open connected subset of $\mathbb{R}^n$, $B$, Stein and Weiss [SW] obtained an important representation theorem for the space $H^2(T_B)$ (Theorem 2.3, Page 93). To have richer results, however, the types of the base $B$ of the tube $T_B$ should be restricted. In the same book they prove more results for the case where tubes are based on open cones, but still for $p = 2$. 
We will continue to study $H^p(T_\Gamma)$ but for the range $1 \leq p \leq \infty$, where $\Gamma$ is an open cone satisfying

(1) $0$ does not belong to $\Gamma$;

(2) Whenever $x, y \in \Gamma$, and $\alpha, \beta > 0$, then $\alpha x + \beta y \in \Gamma$.

The dual cone of $\Gamma$ is defined as the following:

$$\Gamma^* = \{ y \in \mathbb{R}^n : y \cdot x \geq 0, \text{ for any } x \in \Gamma \}.$$

We say that the cone $\Gamma$ is regular if the interior of its dual cone $\Gamma^*$ is nonempty.
Let $\Gamma$ be a regular open cone of $\mathbb{R}^n$. Define

$$K(z) = \int_{\Gamma^*} e^{2\pi i z \cdot t} \, dt, \quad z \in T_\Gamma,$$

and

$$P(x, y) = |K(z)|^2 / K(2iy).$$

One can show that they are, respectively, the Cauchy and Poisson kernels of the tube domain.
Let $\Gamma$ be a regular open cone in $\mathbb{R}^n$. Let $1 \leq p \leq 2$ and $F(x) \in L^p(\mathbb{R}^n)$. Then $F(x)$ is the boundary limit function of $F(x + iy) \in H^p(T_\Gamma)$ if and only if $\text{supp} \hat{F} \subset \Gamma^*$. Moreover, if the condition is satisfied, then

$$F(z) = \int_{\mathbb{R}^n} \chi_{\Gamma^*}(t) e^{2\pi i z \cdot t} \hat{F}(t) \, dt$$

$$= \int_{\mathbb{R}^n} F(t) K(z - t) \, dt$$

$$= \int_{\mathbb{R}^n} P(x - t, y) F(t) \, dt.$$
Let $\Gamma$ be a regular open cone in $\mathbb{R}^n$, and $F \in L^p(\mathbb{R}^n)$, $2 < p \leq \infty$. Then $F(x)$ is the boundary limit of some $F(z) \in H^p(T_\Gamma)$ and as the tempered holomorphic distribution represented by the function $F(z)$ if and only if

d-supp $\hat{F} \subset \Gamma^*$, i.e. $(\hat{F}, \varphi) = 0$, for all $\varphi \in S(\mathbb{R}^n)$ with
\[ \text{supp} \varphi \subset (\Gamma^*)^c = \{ x \in \mathbb{R}^n : x \not\in \Gamma^* \}. \]
Theorem 3

Let $\Gamma$ be a regular open cone in $\mathbb{R}^n$. If $F(x) \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then $F(x)$ is the boundary limit function of some $F(x + iy) \in H^p(T_\Gamma)$ if and only if $\text{d-supp} \hat{F} \subset \Gamma^*$. In such case,

(1) for $1 \leq p \leq 2$,

$$F(z) = \int_{\mathbb{R}^n} \chi_{\Gamma^*}(t) e^{2\pi i z \cdot t} \hat{F}(t) \, dt = \int_{\mathbb{R}^n} F(t) K(z - t) \, dt;$$

(2) for $1 \leq p \leq \infty$,

$$F(z) = \int_{\mathbb{R}^n} P(x - t, y) F(t) \, dt.$$

Moreover, if $\Gamma$ is the first octant of $\mathbb{C}^n$, or even polygonal cone, we still obtain

(3) for $2 < p < \infty$,

$$F(z) = \int_{\mathbb{R}^n} F(t) K(z - t) \, dt.$$
Part II. Section 2.4 Hardy Spaces on Tubes (the Several Complex Variables Setting): $0 \leq p \leq 1$ by G-T. Deng, H-C. Li and T. Qian

Methodology: Rational Approximation in the Several Complex Variables Setting.
\[ \hat{f} = \chi_+ \hat{f}, \quad \chi_+(\xi) = \frac{1}{2} \left( 1 + i \frac{\xi}{|\xi|} \right), \quad \xi = \xi_1 e_1 + \cdots \xi_n e_n. \]

Methodology: Alan McIntosh’s characteristic functions \( \chi_+ \) and \( \chi_- \) and density arguments with Clifford distribution and rational approximation.