7. ORDINARY DIFFERENTIAL EQUATIONS

§7.1. Introduction

A differential equation is one that involves one or more derivatives. With an ordinary differential equation (ODE for short) there are two variables, usually $a$ and $y$ and the derivatives will be $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ...

Where there are three or more variables, such as $x$, $y$, and $z$, the equations involve partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ as well as higher order partial derivatives. When you see the phrase partial differential equation (PDE for short) you will know that it refers to these more difficult equations. The term “ordinary” simply distinguishes the ones that we will be studying from the partial ones.

So an ODE involves just ordinary derivatives (as opposed to partial ones) and they have the general form

$$F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^n y}{dx^n}) = 0.$$ 

The highest order or derivative that occurs is called the order of the ODE. In this chapter we are concentrating on order 1 ODEs. These will have the form

$$F(x, y, \frac{dy}{dx}) = 0.$$ 

Solving an ODE means finding a function $f(x)$ such that if $y = f(x)$ then the ODE is satisfied. And just as we were not able to provide a general method for integrating any given function, so we cannot provide a general method for solving ODEs – not even these degree 1 ODEs. In fact solving ODEs involves two potential difficulties. Solving an ODE boils down to carrying out an integration (sometimes more than one) and it is possible that the integration we have to perform involves a function whose integral cannot be expressed in terms of elementary functions, like powers of $x$, exponential functions, log functions and trig functions.

But it can happen that we can’t even get to the stage where an integration is all that is required. So we develop a bag of “tricks” – techniques that work for special cases.

The simplest type of ODE to solve (provided we can perform the integration) is one of the form

$$\frac{dy}{dx} = f(x).$$

The solution is simply

$$y = \int f(x) \, dx.$$
Example 1: Solve the ODE

\[
\frac{dy}{dx} = \frac{1}{1 + x^2}.
\]

**Solution:** \( y = \int \frac{dx}{1 + x^2} = \tan^{-1}x + c. \)

The solution, \( y = \tan^{-1}x + c \) involves an arbitrary constant. This is a feature of all ODEs. Sometimes the arbitrary constant is of the “plus c” variety. But more usually the constant is buried within the expression that gives the solution.

Example 2: Solve the ODE

\[
\frac{dy}{dx} = 3y.
\]

**Solution:** This doesn’t have the form \( \frac{dy}{dx} = f(x) \), but we can turn the \( \frac{dy}{dx} \) upside down to get

\[
\frac{dx}{dy} = \frac{1}{3y}.
\]

Now make sure you realise that it is not simply a case that \( \frac{dy}{dx} \) is a fraction involving two quantities \( dy \) and \( dx \). No, it is not a fraction, but rather the limit of a fraction. However we proved in an earlier chapter that \( \frac{dx}{dy} = \frac{1}{dy/dx} \), provided that \( dy/dx \neq 0 \). That’s the beauty of the Leibniz notation. Things work as if \( \frac{dy}{dx} \) is a fraction.

So we have an ODE of the type that just requires an integration, except for the fact that the x’s and y’s have been interchanged.

\[
\text{So } x = \frac{1}{3} \int \frac{1}{y} dy
\]

\[
= \frac{1}{3} \log y + c, \text{ for some arbitrary } c.
\]

This gives \( 3x - 3c = \log y \). Now remember that logs are powers so we can write this as

\[
y = e^{3x-3c}.
\]

Now put \( A = e^{-3c} \) and this becomes \( y = Ae^{3x} \). Here \( A \) becomes the arbitrary constant. Well, its not quite arbitrary. The fact that \( A = e^{-3c} \) means that \( A > 0 \). Sometimes the the “arbitrary constant has to be in some interval.
§7.2. Separable ODEs

So we can solve ODEs of the form \( \frac{dy}{dx} = f(x) \) and those of the form \( \frac{dy}{dx} = f(y) \). These are special cases of what is called the separable ODE.

A separable ODE is one of the form \( \frac{dy}{dx} = f(x)g(y) \). It is called “separable” because the derivative can be separated into factors, one of which is a function of \( x \) and the other a function of \( y \).

If \( \frac{dy}{dx} = f(x)g(y) \) we write it as \( \frac{dy}{g(y)} = f(x) \, dx \) and hence

\[
\int \frac{dy}{g(y)} = \int f(x) \, dx.
\]

As usual the intermediate line suggests that \( dx \) and \( dy \) have a separate existence, which they do not. However we have seen, in a previous chapter, how in the context of integrals this leads to correct results.

Example 3: Solve the ODE \( \frac{dy}{dx} = \frac{3x^2}{9y^2 + 4y - 7} \).

Solution: \( \int (9y^2 + 4y - 7) \, dy = \int 3x^2 \, dx \).

Hence \( 3y^2 + 2y^2 - 7y = x^3 + C \) for some arbitrary \( C \).

Now you might think that each integration requires its own arbitrary constant and that we should be writing \( 3y^3 + 2y^2 - 7y + C_1 = x^3 + C_2 \). That’s true, but then we would simply write \( C = C_1 - C_2 \).

We can’t write \( y \) explicitly in terms of \( x \) here. This is often the case. Our solution generally appears in the form \( \phi(x, y) = 0 \) rather than in the form \( y = \phi(x) \).

§7.3. Homogeneous ODEs

A homogeneous ODE is one of the form \( \frac{dy}{dx} = f(x, y) \) where \( f(kx, ky) = f(x, y) \) for all \( x, y \) and \( k \). In such a case we can write \( f(x, y) = f(1, y/x) \). So if we put \( v = \frac{y}{x} \) the ODE takes on the form \( \frac{dy}{dx} = f(1, v) \), which we can write as \( \phi(v) \). So we have to solve the ODE

\[
\frac{dy}{dx} = \phi(v).
\]

We’ve now got 3 variables, but we can write \( y = vx \), so we can express \( y \) in terms of \( v \) and \( x \) in this way to get an ODE in terms of just \( v \) and \( x \).

Now \( \frac{dy}{dx} = \frac{d(vx)}{dx} = v + x \frac{dv}{dx} \), but this is \( \phi(v) \).

So \( x \frac{dv}{dx} = \phi(v) - v \).
Hence \( \int \frac{dx}{x} = \int [\phi(v) - v] \, dx \).

Hence \( \log x = \int [\phi(v) - v] \, dx \) and so

\[ x = e^{\int [\phi(v) - v] \, dx}. \]

We carry out the integration, if we can, getting a function of \( v \), together with an arbitrary constant. Then we replace \( v \) by \( \frac{y}{x} \) to express the solution in terms of just \( x \) and \( y \).

[Do not attempt to remember any of these formulae. Just remember to put \( v = \frac{y}{x} \).]

**Example 4:** Solve the ODE \( \frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 + xy} \).

**Solution:** Clearly this is a homogeneous ODE.

Put \( v = \frac{y}{x} \), then \( y = vx \).

\[ \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}. \]

\[ \therefore v + x \frac{dv}{dx} = \frac{y^2 - 2xy}{x^2 + xy} = \frac{v^2 x^2 - 2vx^2}{x^2 + vx^2} = \frac{v^2 - 2v}{1 + v} \]

Hence \( x \frac{dv}{dx} = \frac{v^2 - 2v}{1 + v} - v \)

\[ = \frac{v^2 - 2v - v - v^2}{1 + v} \]

\[ = \frac{-3v}{1 + v}. \]

So \( \int \frac{1 + v}{v} \, dv = -3 \int \frac{dx}{x} \).

\[ \therefore \log v + v = -3 \log x + C, \text{ for some arbitrary constant } C \]

\[ \therefore \log v + \log e^v = \log \frac{1}{x^3} + \log A, \text{ for some } A \]

\[ \therefore \log ve^v = \log \frac{A}{x^3} \]

\[ \therefore ve^v = \frac{A}{x^3} \]

\[ \therefore \frac{y}{x} e^{\frac{y}{x}} = \frac{A}{x^3} \]

\[ \therefore x^2 y e^{\frac{y}{x}} = A. \]
§7.4. Linear ODEs

A linear ODE is one of the form
\[ \frac{dy}{dx} + P(x)y = Q(x) \]
for some functions of x P(x) and Q(x). The technique for solving these ODEs is to multiply through by some function of x, \( \mu(x) \). This will give
\[ \mu(x) \frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \]
........................... (1)
The function \( v = \mu(x) \) is called an integrating factor. Then
\[ v \frac{dy}{dx} + vP(x)y = vQ(x) \]
........................... (2)
But not any old function of x will do. We want to choose \( v \) so that the left hand side of the equation (2) becomes \( \frac{d(vy)}{dx} \). Then all we would have to do would be to integrate both sides of
\[ \frac{d(vy)}{dx} = \mu(x)Q(x) \]
to get \( vy = \int \mu(x)Q(x) \, dx \) and so
\[ y = \frac{1}{\mu(x)} \left( \int \mu(x)Q(x) \, dx + C \right) \]
would be the solution.

So what should \( \mu(x) \) be? Let \( v = \mu(x) \).

Now \( \frac{d(vy)}{dx} = v \frac{dy}{dx} + y \frac{dv}{dx} \) so comparing this with the left hand side of (2) we would need
\[ \frac{dv}{dx} = vP(x) \]
Hence \( \int \frac{dv}{v} \, dv = \int P(x) \, dx \).
\[ \therefore \log v = \int P(x) \, dx \] and so
\[ \mu(x) = v = e^{\int P(x) \, dx} \]
There could be some arbitrary constant here but since we just want any suitable v we may as well take this constant to be zero.

So the solution is \( y = \frac{1}{\mu(x)} \left( \int \mu(x)Q(x) \, dx + C \right) \) where \( \mu(x) = e^{\int P(x) \, dx} \).

That is, the solution is \( y = e^{-\int P(x) \, dx} \left( \int e^{\int P(x) \, dx}Q(x) \, dx + C \right) \).
You might be tempted to memorise this formula. It may be worth remembering that the integrating factor is \( e^{\int P(x) \, dx} \) (provided you can remember that \( P(x) \) is the coefficient of \( y \)). But beyond that my advice is to remember the method. That is less likely to lead to errors.

**Example 5:** Solve the ODE \( \frac{dy}{dx} + \frac{2}{x} y = 12x^3 \).

**Solution:** \( \int \frac{2}{x} \, dx = \log x \) so the integrating factor is \( e^{2 \log x} = x^2 \).

So \( x^2 \frac{dy}{dx} + 2xy = 12x^5 \).

\[ \therefore \quad \frac{d(x^2 y)}{dx} = 12x^5 \]

\[ \therefore \quad x^2 y = \int 12x^5 \, dx = 2x^6 + C \]

\[ \therefore \quad y = 2x^4 + \frac{C}{x^2} \] for arbitrary \( C \).

§7.5. Initial Conditions

A first order differential equation will have one arbitrary constant in its solution. But if we are given an additional piece of information we can use this to determine the value of this constant and so obtain a particular solution.

If the equation represents the position of a particle moving along a straight line we might represent its distance from a fixed point by \( x \) and the elapsed time by \( t \) (using appropriate units). Since \( \frac{dx}{dt} \) is the velocity a first order differential equation, of the form

\[ F(x, t, \frac{dx}{dt}) = 0 \]

will relate the position and speed with time.

Very often the extra piece of information that enables us to find the value of the constant is the initial position, that is the value of \( x \) when \( t = 0 \). Such information would be called an **initial condition**.

With a second order differential equation

\[ F(x, t, \frac{dx}{dt}, \frac{d^2 x}{dt^2}) = 0 \]

the second derivative represents acceleration. There will be two arbitrary constants and if we are to find their values we need two pieces of extra information. Often these will be the initial position and velocity, that is the values of \( x \) and \( \frac{dx}{dt} \) when \( t = 0 \). These are called the **initial conditions**.

We often use the phrase “initial conditions” even when the additional information is not for \( t = 0 \) and even when time is not one of the variables. This is an abuse of language but
we know what we mean. It is preferable to having to make up some other phrase to cover the more general situation.

Sometime we call the initial conditions, boundary conditions. This terminology derives from the situation with partial differential equations. If there are 3 variables, x, y and z we might have a boundary within in which there is some surface (perhaps in more than three dimensions) that is described by the PDE. The initial conditions in this case might be the equation of the boundary.

**Example 6:** Solve the ODE
\[
\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \quad \text{if } x = 1 \text{ when } x = 0.
\]

**Solution:**
\[
x = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + c.
\]

Since \( x = 1 \) when \( t = 0 \), \( c = 1 \).

Hence the solution is \( x = 1 + \sin^{-1} t \).

**Example 7:** Solve the ODE
\[
\frac{dx}{dt} = x^2 \quad \text{if } x = 2 \text{ when } t = 0
\]

**Solution:**
\[
\frac{dt}{dx} = \frac{1}{x^2}.
\]

So \( t = \int \frac{1}{x^2} \, dx \)
\[
= -\frac{1}{x} + c, \text{ for some arbitrary } c.
\]

Since \( x = 2 \) when \( t = 0 \) we have \( c = \frac{1}{2} \) and so the solution is
\[
t = -\frac{1}{x} + \frac{1}{2} = \frac{x - 2}{2x}.
\]

\[
\therefore 2tx = x - 2.
\]

\[
\therefore x(1 - 2t) = 2.
\]

\[
\therefore x = \frac{2}{1 - 2t}.
\]

Note that we haven’t finished when we have found the constant. We must express the dependent variable, in this case \( x \), in terms of the independent variable, in this case \( t \).

**Example 8:** Solve the ODE
\[
\frac{dx}{dt} = \frac{e^t}{\sqrt{x}} \quad \text{if } x = 4 \text{ when } t = 0.
\]

**Solution:**
\[
\int \sqrt{x} \, dx = \int e^t \, dt.
\]

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Hence \( \frac{2}{3} x^{3/2} = e^t + C \) for some arbitrary \( C \).

Since \( x = 9 \) when \( t = 0 \), we have \( C = 17 \).

Hence \( \frac{2}{3} x^{3/2} = e^t + 17 \).

\[ \therefore x^{3/2} = \frac{3}{2} (e^t + 17). \]

\[ \therefore x = \left[ \frac{3}{2}(e^t + 17) \right]^{2/3} \]

**Example 9:** Solve the ODE 

\[ \frac{dx}{dt} = \frac{3t - x}{t + x} \]

if \( x = 2 \) when \( t = 1 \).

**Solution:** Clearly this is a homogeneous ODE.

Put \( v = \frac{x}{t} \), then \( x = vt \).

\[ \therefore \frac{dx}{dt} = v + t \frac{dv}{dt} \].

\[ \therefore v + t \frac{dv}{dt} = \frac{3t - x}{t + x} \]

\[ \therefore v + t \frac{dv}{dt} = \frac{3t - vt}{t + vt} \]

\[ \therefore v + t \frac{dv}{dt} = \frac{3 - v}{1 + v} \]

Hence \( t \frac{dv}{dt} = \frac{3 - v}{1 + v} - v \)

\[ = \frac{3 - v - v - v^2}{1 + v} \]

\[ = \frac{v^2 + 2v - 3}{1 + v} \]

So \( \int \frac{v + 1}{v^2 + 2v - 3} \, dv = -\int \frac{dt}{t} = -\log t + C \)

Now \( \frac{v + 1}{v^2 + 2v - 3} = \frac{v + 1}{(v + 3)(v - 1)} \).

We express this using partial fractions.

Write \( \frac{v + 1}{(v + 3)(v - 1)} = \frac{A}{v + 3} + \frac{B}{v - 1} \).

\( \therefore v + 1 = A(v - 1) + B(v + 3) \).

Putting \( v = 1 \) we get 2 = 4B, so \( B = \frac{1}{2} \).

Putting \( v = -3 \) we get -2 = -4A so \( A = -\frac{1}{2} \).

Hence \( \int \frac{v + 1}{v^2 + 2v - 3} \, dv = -\frac{1}{2} \int \frac{dv}{v + 3} + \frac{1}{2} \int \frac{dv}{v - 1} \).
\[
= -\frac{1}{2} \log(v + 3) + \frac{1}{2} \log(v - 1)
\]
\[
= \log \sqrt{\frac{v - 1}{v + 3}}.
\]
Hence \(\log \sqrt{\frac{v - 1}{v + 3}} = C - \log t\).

When \(t = 1\), \(x = 2\) and so \(v = 2\).

\[\therefore \log \sqrt{\frac{1}{3}} = C\]

\[\therefore C = -\frac{1}{2} \log 5.\]

Hence the solution is

\[\log \sqrt{\frac{v - 1}{v + 3}} = -\frac{1}{2} \log 5 - \log t\]
\[= \log \frac{1}{\sqrt{5t}}.\]

Hence \(\sqrt{\frac{v - 1}{v + 3}} = \frac{1}{\sqrt{5t}}\).

\[\therefore \frac{v - 1}{v + 3} = \frac{1}{5t^2}.\]

\[\therefore \frac{x - t}{x + 3t} = \frac{1}{5t^2}.\]

\[\therefore \frac{x + 3t}{x - t} = 5t^2.\]

\[\therefore x + 3t = 5t^2(x - t).\]

\[\therefore x(5t^2 - 1) = 5t^2 + 3t.\]

\[\therefore x = \frac{5t^2 + 3t}{5t^2 - t}.\]

**Example 10:** Solve the ODE

\[
\frac{dx}{dt} + \frac{x}{t + 1} = t \quad \text{if} \quad x = 0 \text{ when } t = 1.
\]

**Solution:** \(\int \frac{1}{t + 1} dt = \log (t + 1)\) so the integrating factor is \(e^{\log(t+1)} = t + 1\).

So \((t + 1)\frac{dx}{dt} + x = t^2 + t.\)

\[\therefore \frac{d(x(t + 1))}{dx} = t^2 + t.\]

\[\therefore x(t + 1) = \int (t^2 + t) \ dt
\]
\[= \frac{1}{3} t^3 + \frac{1}{2} t^2 + C\]
Since \( x = 0 \) when \( t = 1 \), \( C = -\frac{5}{6} \).

\[
\therefore x(t + 1) = \frac{1}{3} t^3 + \frac{1}{2} t^2 - \frac{5}{6}
= \frac{2t^3 + 3t^2 - 5}{6}
\]

\[
\therefore x = \frac{2t^3 + 3t^2 - 5}{6(t + 1)}.
\]

§7.6. Exact Equations

A first order differential equation of the form \( P(x, y) + Q(x, y) \frac{dy}{dx} = 0 \) is called **exact**

if \( \frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}. \)

**Theorem 1:** If \( \frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x} \) then there exists \( R(x, y) \) such that

\[
\frac{\partial R(x, y)}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial R(x, y)}{\partial y} = Q(x, y).
\]

**Proof:** We omit the proof.

So if \( P(x, y) + Q(x, y) \frac{dy}{dx} = 0 \) is exact then there exists \( R(x, y) \) such that

\[
\frac{\partial R(x, y)}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial R(x, y)}{\partial y} = Q(x, y).
\]

If the solution is \( y = f(x) \) then \( R(x, y) \) is just a function of \( x \) and

\[
\frac{d}{dx} R(x, y) = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial y} \frac{dy}{dx}
= P(x, y) + Q(x, y) \frac{dy}{dx}
= 0,
\]

so \( R(x, y) = C \) defines \( y \) implicitly as a function of \( x \), for arbitrary \( C \).

**Example 11:** Show that the ODE \( 9x^2 + 4xy + (2x^2 + 3y^2) \frac{dy}{dx} = 0 \) is exact, and hence solve it.

**Solution:** Let \( P(x, y) = 9x^2 + 4xy \) and \( Q(x, y) = 2x^2 + 3y^2 \).

Then \( \frac{\partial P}{\partial y} = 4x = \frac{\partial Q}{\partial x}. \)

By **Theorem 1** there exists a function \( R(x, y) \) such that

\[
\frac{\partial R}{\partial x} = 9x^2 + 4xy \quad \text{and} \quad \frac{\partial R}{\partial y} = 2x^2 + 3y^2.
\]
Integrating \( \frac{\partial R}{\partial x} \) with respect to \( x \) we get \( R(x, y) = 3x^3 + 2x^2y + C(y) \). Here the “constant of integration” is a constant as far as \( x \) is concerned, but in fact it is a function just of \( y \).

Then \( \frac{\partial R}{\partial y} = 2x^2 + \frac{dC}{dy} = Q(x, y) = 2x^2 + 3y^2 \).

Hence \( \frac{dC}{dy} = 3y^2 \).

Thus \( C(x, y) = y^3 + K \) for some constant \( K \).

Hence \( R(x, y) = 3x^3 + 2x^2y + y^3 + K \) and the solution can be expressed implicitly as

\[
3x^3 + 2x^2y + y^3 + K = C,
\]

in other words

\[
3x^3 + 2x^2y + y^3 = K - C.
\]

Clearly \( K - C \) is simply an arbitrary constant, which we could have written as \( C \). In other words, when integrating \( \frac{dC}{dy} \) we are permitted to ignore the arbitrary constant.

The implicit solution is

\[
3x^3 + 2x^2y + y^3 = C
\]

for arbitrary \( C \).

§7.7. Higher Order ODEs

An \( n \)th order ODE involves derivatives up to \( \frac{d^ny}{dx^n} \). The context in which to discuss the solutions to higher order ODEs is abstract vector spaces. You may not yet have studied these so here is a brief account that will be sufficient for our purposes here.

We know that vectors are things like \( (x, y, z) \) that can represent points in space. You may not yet know that functions of a real variable can be viewed as functions. What is needed for vectors is a way of adding two together and multiplying one of them by a scalar. In this cases the scalars will be the real numbers. There is a whole bunch of axioms, but we won’t bother discussing them here.

The functions \( f(x) = \sin x \) and \( g(x) = \cos x \) can be considered as vectors. We can add them: \( f(x) + g(x) = \sin x + \cos x \). We can also multiply them by real numbers: \( 2f(x) = 2\sin x \). But don’t vectors have to have components? Not really, but you could think of the values of the function as the components, in which case we have infinitely many!

A linear combination of the functions \( f_1(x), f_2(x), \ldots, f_n(x) \) is a function of the form

\[
a_1f_1(x) + a_2f_2(x) + \ldots + a_nf_n(x).
\]

Example 12: \( 3\sin x - 5\cos x \) is a linear combination of \( \sin x \) and \( \cos x \).

Example 13: The constant function \( f(x) = 1 \) is a linear combination of the functions \( g(x) = \sin^2x \) and \( \cos^2x \) since \( \sin^2x + \cos^2x = 1 \).

The zero function \( z(x) = 0 \) is a linear combination of the functions \( f(x) = 1 \), \( g(x) = \sin^2x \) and \( h(x) = \cos^2x \) since

\[
(-1)1 + 1.\sin^2x + 1.\cos^2x = 0.
\]
Of course the zero function $z(x) = 0$ is always a linear combination of any set of functions. In the above example we could have written

$$0.1 + 0.\sin^2x + 0.\cos^2x = 0.$$ 

But we call such a linear combination, where all the coefficients are zero, a trivial linear combination. If there is a non-trivial linear combinations of $f_1(x), f_2(x), ..., f_n(x)$ we say that the functions $f_1(x), f_2(x), ..., f_n(x)$ are linearly dependent. In this case one of the functions can be expressed as a linear combination of the others.

**Example 14:** The functions $\sin^2x, \cos^2x$ and $1$ are linearly dependent since $\sin^2x + \cos^2x = 1$. We can therefore write $\cos^2x = 1 - \sin^2x$.

If $f_1(x), f_2(x), ..., f_n(x)$ are not linearly dependent we say that they are linearly independent. In that case none of the functions are a linear combination of the others.

**Example 15:** Show that the following functions are linearly independent: $f(x) = \sin x, g(x) = \cos x, h(x) = x$.

**Solution:** It’s no good saying “I can’t think of any connection between these three functions.” A proof is required. We do this as a Proof By Contradiction. Suppose $a.\sin x + b.\cos x + cx = 0$ for some real numbers $a, b, c$.

Now this is not an equation to be solved for certain $x$. It is a relationship between three functions and therefore holds for all $x$.

Put $x = 0$. Then $b = 0$.

Put $x = \pi$. Then $c\pi = 0$, so $c = 0$.

Put $x = \pi/2$. Then $a = 0$.

So there are no non-trivial linear combinations that equal the zero function and hence these functions are linearly independent.

§7.8. Linear ODEs With Constant Coefficients

An $n^{th}$ order linear ODE with constant coefficients is one of the form

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0y = Q(x)$$

where the $a_i$’s are constants.

The associated homogeneous ODE is

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0y = 0.$$ 

There are two fundamental facts about linear ODEs with constant coefficients. We state them here without proof.

**Theorem 2:** (1) The general solution to a linear ODE with constant coefficients has the form $y = H(x) + P(x)$ where $H(x)$ is the general solution to the associated homogeneous ODE and $P(x)$ is any particular solution to the original ODE.

(2) There are $n$ linearly independent solutions to a homogeneous $n^{th}$ order linear ODE, but no more.
This means that if we have to solve the ODE
\[
\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = Q(x).
\]
we find just one solution, \(g(x)\) to this equation and \(n\) linearly independent solutions \(f_1(x), f_2(x), ..., f_n(x)\) to the homogeneous ODE
\[
\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0.
\]
The general solution to the original ODE will be
\[
y = A_1f_1(x) + A_2f_2(x) + ... + A_nf_n(x) + g(x).
\]
There will be as many arbitrary constants in the solution as the order of the ODE.

§ 7.9. The Homogeneous Case

We define the characteristic equation of the homogeneous linear ODE with constant coefficients
\[
\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0
\]
to be the polynomial
\[
\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0
\]
Here the symbol \(\lambda\) is an indeterminate and represents a complex number.

**Theorem 3:** Suppose \(\lambda\) is a real solution to the characteristic equation. Then \(y = e^{\lambda x}\) is a solution to the homogeneous ODE.

**Proof:** Let \(y = e^{\lambda x}\).

Then
\[
\frac{dy}{dx} = \lambda e^{\lambda x},
\]
\[
\frac{d^2 y}{dx^2} = \lambda^2 e^{\lambda x},
\]
\[
\vdots
\]
\[
\frac{d^n y}{dx^n} = \lambda^n e^{\lambda x}.
\]
Hence
\[
\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = (\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0)e^{\lambda x} = 0.
\]

**Example 16:** Solve the ODE \(\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} = 21y\) if \(\frac{dy}{dx} = -5\) and \(y = 5\) when \(x = 0\).

**Solution:** Writing the ODE as \(\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - 21y = 0\), the characteristic equation is
\[
\lambda^2 + 3\lambda - 21 = 0.
\]
This factorises as \((\lambda - 3)(\lambda + 7) = 0\), giving solutions \(\lambda = 3, -7\).
Hence the general solution is \(y = Ae^{3x} + Be^{-7x}\).

Hence \(\frac{dy}{dx} = 3Ae^{3x} - 7Be^{-7x}\).
Using the initial conditions we get
\[
\begin{align*}
A + B &= 5 \\
3A - 7B &= -5
\end{align*}
\]
Solving, we get \( A = 3 \) and \( B = 2 \).
Hence the solution is \( y = 3e^{3x} + 2e^{-7x} \).

This leaves the case of repeated solutions and non-real solutions. We set out the full story in the following theorem that we do not prove.

**Theorem 4:** Suppose the characteristic equation of a homogeneous linear ODE (with constant coefficients) has the following solutions:

- **Real Solutions:** \( \lambda_1, ..., \lambda_t \), with \( \lambda_i \) having multiplicity \( m_i \).
- **Non-Real Solutions:** \( \alpha_1 \pm \beta_1i, ..., \alpha_s \pm \beta_s i \), with \( \alpha_t \pm \beta_t i \) having multiplicity \( n_t \).

Then the homogeneous ODE has the general solution
\[
y = A_1(x)e^{\lambda_1 x} + ... + A_t(x)e^{\lambda_t x} + e^{\alpha_1 x}[B_1(x) \cos(\beta_1 x) + C_1(x) \sin(\beta_1 x)] + ... + e^{\alpha_s x}[B_s(x) \cos(\beta_s x) + C_s(x) \sin(\beta_s x)]
\]
where \( A_t(x) \) is an arbitrary polynomial of degree \( r_t - 1 \) and \( B_t(x) \) and \( C_t(x) \) are arbitrary polynomials of degree \( s_t - 1 \).

**Example 17:** Solve the ODE \( \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0 \) if \( \frac{dy}{dx} = -7 \) and \( y = 9 \) when \( x = 0 \).

**Solution:** The characteristic equation is \( \lambda^2 - 2\lambda + 5 = 0 \).
Using the quadratic equation formula we get \( \lambda = -1 \pm 2i \).
Hence the general solution is \( y = e^{-x}[A \cos 2x + B \sin 2x] \).
Hence \( \frac{dy}{dx} = e^{-x}[2A \sin 2x + 2B \cos 2x] - e^{-x}[B \cos 2x + A \sin 2x] = e^{-x}[(2B - A)\cos 2x - (2A + B)\sin 2x] \)
When \( x = 0 \), \( y = A \) and \( \frac{dy}{dx} = 2B - A \).
Using the initial conditions we get \( A = 9 \) and \( 2B - A = -7 \), that is, \( A = 9 \) and \( B = 1 \).
Hence the solution is \( y = e^{-x}[9 \cos 2x + \sin 2x] \).

**Example 18:** Find the general solution to a linear ODE with constant coefficients if the characteristic equation has solutions:
\[
3, \ 3, \ 3, \ 5, \ 2 \pm 3i, \ 2 \pm 3i.
\]

**Solution:** The general solution will be:
\[
y = (A + Bx + Cx^2)e^{3x} + De^{5x} + e^{2x}[(E + Fx)\cos 3x + (G + Hx)\sin 3x].
\]
§7.10. Finding Particular Solutions

There are many methods given for finding a particular solution to a non-homogeneous ODE (linear with constant coefficients)
\[
d^n y/a_n + a_{n-1}d^{n-1}y/dx + ... + a_1dy/dx + a_0y = Q(x).
\]
None of them works in all cases but they all provide a particular solution when Q(x) has terms that are polynomials, exponential functions, the sine and cosine functions and products of these.

The simplest method is a sophisticated form of “trial and error”. Take each term in Q(x) and differentiate it repeatedly. If you get to a stage where the original term, and its successive derivatives are linearly dependent, you are in luck. You simply take a general linear combination of these functions. You then substitute it into the ODE and find the correct coefficients in the linear combination to get a solution.

This probably sounds very vague. Indeed it is only when you have seen many examples that you will come to fully understand the technique.

Example 19: Find a particular solution to the ODE
\[
\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = -2x^2.
\]

Solution: Differentiating \(-2x^2\) gives \(-4x\). The second derivative will be \(-4\). The third derivative is 0 and we have reached linear dependence because
\[
\frac{d^3y}{dx^3} = 0, \quad \frac{d^2y}{dx^2} = 0, \quad \frac{dy}{dx} = 0, \quad y = 0
\]
is a non-trivial linear combination that is zero.

So we take as a “trial” solution a general quadratic
\[
y = ax^2 + bx + c.
\]
Then
\[
\frac{dy}{dx} = 2ax + b \quad \text{and} \quad \frac{d^2y}{dx^2} = 2a.
\]
Substituting into the ODE we get
\[
(2a) + 5(2ax + b) - 6(ax^2 + bx + c) = -2x^2.
\]
\[
\therefore (6a - 2)x^2 + (6b - 10a)x + (6c - 5b - 2a) = 0.
\]
Since the functions \(x^2\), \(x\) and 1 are linearly independent (easily shown) this must be the trivial linear combination, and so
\[
6a - 2 = 0
\]
\[
6b - 10a = 0
\]
\[
6c - 5b - 2a = 0.
\]
Solving this we get
\[
a = \frac{1}{3}, \quad b = \frac{5}{9}, \quad c = \frac{31}{54}.
\]
Hence a particular solution is \( y = \frac{1}{3} x^2 + \frac{5}{9} x + \frac{31}{54} \).

**Example 20:** Find the solution to the ODE \( \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 6y = -2x^2 \) if 

\[
y(0) = 0 \text{ and } \frac{dy}{dx} = \frac{1}{2} \text{ when } x = 0.
\]

**Solution:** The associated homogeneous ODE is \( \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 6y = 0 \), which has characteristic equation \( \lambda^2 + 5\lambda - 6 = 0 \).

\[
\lambda = -6 \text{ or } 1.
\]

The general solution to the homogeneous ODE is \( y = Ae^{-6x} + Be^x \) and so the general solution to the non-homogeneous one is

\[
y = Ae^{-6x} + Be^x + \frac{1}{3} x^2 + \frac{5}{9} x + \frac{31}{54}.
\]

\[
\frac{dy}{dx} = -6Ae^{-6x} + Be^x + \frac{2}{3} x + \frac{5}{9}.
\]

When \( x = 0 \), \( y = A + B + \frac{31}{54} \) and \( \frac{dy}{dx} = -6A + B + \frac{5}{9} \).

Hence

\[
\begin{align*}
A + B + \frac{31}{54} &= 0 \\
-6A + B + \frac{5}{9} &= \frac{1}{2}.
\end{align*}
\]

This gives \( A = -\frac{2}{27} \) and \( B = -\frac{1}{2} \).

The solution is thus \( y = -\frac{2}{27} e^{-6x} - \frac{1}{2} e^x + \frac{1}{3} x^2 + \frac{5}{9} x + \frac{31}{54} \)

\[
= \frac{1}{54} \left[ -4e^{-6x} - 27e^x + 18x^2 + 30x + 31 \right].
\]

**Example 21:** Find a particular solution to the ODE \( \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 6y = 16e^{2x} \).

**Solution:** The derivative of \( 16e^{2x} \) is \( 32e^{2x} \) so we get linear dependence.

So our trial solution should be \( y = ae^{2x} \).

\[
\frac{dy}{dx} = 2ae^{2x} \text{ and } \frac{d^2 y}{dx^2} = 4ae^{2x}.
\]

Substituting into the ODE we get

\[
4ae^{2x} + 10ae^{2x} - 6ae^{2x} = 16e^{2x}
\]

so \( 8a = 16 \) which gives \( a = 2 \).

Hence \( y = 2e^{2x} \) is a particular solution.
Example 22: Find a particular solution to the ODE \( \frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 2e^{-6x} \).

Solution: The derivative of \( 2e^{-6x} \) is \( -12e^{-6x} \) so we get linear dependence. So our trial solution should be \( y = ae^{-6x} \).

\[ \frac{dy}{dx} = -6ae^{-6x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 36ae^{-6x} \]

Substituting into the ODE we get
\[ 36ae^{-6x} - 30ae^{-6x} - 6ae^{-6x} = 2e^{-6x} \]
\[ 0a = 2 \] which has no solution.

This does not mean that the ODE has no solution, simply that we took the wrong “trial” solution. The reason is that the general solution to the associated homogeneous ODE is \( y = Ae^{-6x} + Be^x \) and so adding another \( e^{-6x} \) term will still give 0 on the right hand side.

Without going into why, let me simply say if there is overlap between the homogeneous solution and the “trial” solution, multiply the trial solution by \( x \) until you get a term that is not covered by the homogeneous solution. In this case we try \( y = axe^{-6x} \).

\[ \therefore \frac{dy}{dx} = ae^{-6x} - 6axe^{-6x} \quad \text{and} \]
\[ \frac{d^2y}{dx^2} = -6ae^{-6x} - 6ae^{-6x} + 36axe^{-6x} \]
\[ = -12ae^{-6x} + 36axe^{-6x} \]

Substituting, we get
\[ (-12ae^{-6x} + 36axe^{-6x}) + 5(2ae^{-6x} - 6axe^{-6x}) - 6axe^{-6x} = 2e^{-6x} \]
\[ \therefore ae^{-6x}(-12 + 5) = 2e^{-6x} \]
\[ \therefore -7a = 2 \]
\[ \therefore a = -\frac{2}{7} \]

Hence \( y = -\frac{2}{7}xe^{-6x} \) is a particular solution.

Example 23: Find a particular solution to the ODE \( \frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = -60\sin 2x \).

Solution: The derivative of \( 7\sin 2x \) is \( 14\cos x \) and the second derivative is \( -28\sin 2x \). So we have reached linear dependence. So we try \( y = a\sin 2x + b\cos 2x \).

\[ \therefore \frac{dy}{dx} = 2a\cos 2x - 2b\sin 2x \]
\[ \therefore \frac{d^2y}{dx^2} = -4a\sin 2x - 4b\cos 2x \]

Substituting, we get
\[ (-4a\sin 2x - 4b\cos 2x) + 5(2a\cos 2x - 2b\sin 2x) - 6(a\sin 2x + b\cos 2x) = -60\sin 2x \]
\[ \therefore (-4a - 10b - 6a)\sin 2x + (-4b + 10a - 6b)\cos 2x = -60\sin 2x \]
\[ \therefore 10a + 10b = 60 \quad \text{and} \]
\[ 10a - 10b = 0 \]

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\[ a = b \text{ and } 20a = 60, \text{ so } a = b = 3. \]
The particular solution is therefore \( y = 3 \sin 2x + 3 \cos 2x. \)

**Example 24:** Find a particular solution to the ODE \( \frac{d^2y}{dx^2} + y = \sin x. \)

**Solution:** The derivative of \( \sin x \) is \( \cos x \) and the second derivative is \( -\sin x. \) So we have reached linear dependence. So we might try \( y = a \sin x + b \cos x. \) But wait. We must first consider the homogeneous solution to see if there is any overlap.

The characteristic equation is \( \lambda^2 + 1 = 0, \) which has solutions \( \pm i. \) This means that the general solution to the associated homogeneous solution is \( y = A \sin x + B \cos x. \) This overlaps the right hand side of our non-homogeneous equation and so we must multiply by \( x. \) So we try \( y = a x \sin x + b x \cos x. \)

\[ \therefore \frac{dy}{dx} = a \sin x + a x \cos x + b \cos x - b x \sin x. \]

\[ \therefore \frac{d^2y}{dx^2} = a \cos x + a (\cos x - x \sin x) - b \sin x - b (\sin x + x \cos x) \]
\[ = 2a \cos x - 2b \sin x - a x \sin x - b x \cos x \]

Substituting, we get
\[ 2a \cos x - 2b \sin x - a x \cos x - b x \sin x = \sin x. \]
\[ \therefore a = 0 \text{ and } b = -\frac{1}{2}. \]
The particular solution is \( y = -\frac{1}{2} \sin x. \)

**Example 25:** Find a particular solution to the ODE \( \frac{d^2y}{dx^2} - 6 \frac{dy}{dx} + 9y = e^{3x}. \)

**Solution:** The derivative of \( e^{3x} \) is \( 3e^{3x} \) and so we have reached linear dependence. Our first thought is to try \( y = ae^{3x}. \)

The characteristic equation of the associated homogeneous solution is \( \lambda^2 - 6\lambda + 9 = 0, \) which has repeated solutions \( \lambda = 3, 3. \) This means that the general solution to the associated homogeneous solution is \( y = Ae^x + Bxe^x. \) This overlaps the right hand side of our non-homogeneous equation and so we must multiply by \( x^2. \) (Clearly just multiplying by \( x \) is not sufficient.)

So we try \( y = ax^2e^{3x}. \)

\[ \therefore \frac{dy}{dx} = 2axe^{3x} + 3ax^2e^{3x}. \]

\[ \therefore \frac{d^2y}{dx^2} = 2ae^{3x} + 6axe^{3x} + 6axe^{3x} + 9ax^2e^{3x}. \]
\[ = 2ae^{3x} + 12axe^{3x} + 9ax^2e^{3x}. \]

Substituting, we get
\[ (2ae^{3x} + 12axe^{3x} + 9ax^2e^{3x}) - 6(2axe^{3x} + 3ax^2e^{3x}) + 9ax^2e^{3x} = e^{3x}. \]
\[ \therefore 2ae^{3x} = e^{3x}. \]
\[ \therefore a = \frac{1}{2} \text{ and so the particular solution is } y = \frac{1}{2}x^2e^{3x}. \]
SUMMARY
FIRST ORDER ODEs

<table>
<thead>
<tr>
<th>TYPE</th>
<th>FORM</th>
<th>SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Separable</td>
<td>( \frac{dy}{dx} = P(x)Q(y) )</td>
<td>( \int \frac{dy}{Q} = \int P , dx )</td>
</tr>
</tbody>
</table>
| Homogeneous   | \( \frac{dy}{dx} = P(v) \) where \( v = \frac{y}{x} \) | Integrating Factor: \( \mu(x) = e^{\int P \, dx} \)  
Solution is \( \mu(x)y = \int \mu(x)Q(x) \, dx \) |
| Linear        | \( \frac{dy}{dx} + P(x)y = Q(x) \) | \( \int \frac{dx}{x} = \int \frac{dv}{P(v) - v} \) |
| Exact         | \( \frac{dy}{dx} + \frac{P(x, y)}{Q(x, y)} \), where \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0 \) | Find \( R(x, y) \) so \( \frac{\partial R}{\partial x} = P, \frac{\partial R}{\partial y} = Q \).  
Solution is \( R(x, y) = \text{constant} \) |

LINEAR ODEs WITH CONSTANT COEFFICIENTS

HOMOGENEOUS: \( \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 y = 0 \) (\( a_i \)'s are constants).

Solve the Characteristic Equation: \( \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \).

Real Solution \( \lambda \) with multiplicity \( m \):
\[
(A_1 + A_2 x + \ldots + A_{m-1} x^{m-1})e^{\lambda x}
\]

Non-Real Solution \( \alpha \pm \beta i \) with multiplicity \( m \):
\[
(A_1 + A_2 x + \ldots + A_{m-1} x^{m-1})\cos \beta x + (B_1 + B_2 x + \ldots + B_{m-1} x^{m-1}) \sin \beta x
\]

NON-HOMOGENEOUS: \( \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = Q(x) \).

General Solution to Non-Homogeneous = General Solution to Homogeneous + Particular Solution to Non-Homogeneous

PARTICULAR SOLUTION \( [p(x) \text{ is a polynomial}] \)

<table>
<thead>
<tr>
<th>Q(x)</th>
<th>BASIC TRIAL SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>general polynomial of the same degree</td>
</tr>
<tr>
<td>( e^{\alpha x} )</td>
<td>( a e^{\alpha x} )</td>
</tr>
<tr>
<td>( \sin \alpha x )</td>
<td>( a \sin \alpha x + b \cos \alpha x )</td>
</tr>
<tr>
<td>( \cos \alpha x )</td>
<td>( a \sin \alpha x + b \cos \alpha x )</td>
</tr>
<tr>
<td>( p(x)e^{\alpha x} )</td>
<td>( a(x)e^{\alpha x} \text{ where } a(x) \text{ is a polynomial of the same degree} )</td>
</tr>
<tr>
<td>( p(x) \sin \alpha x )</td>
<td>( a(x) \sin x + b(x) \cos x \text{ where } a(x), b(x) \text{ are polynomials of the same degree} )</td>
</tr>
<tr>
<td>( p(x) \cos \alpha x )</td>
<td>( a(x) \sin x + b(x) \cos x \text{ where } a(x), b(x) \text{ are polynomials of the same degree} )</td>
</tr>
</tbody>
</table>

If a term in the trial solution overlaps with a term in the general solution of the associated homogeneous ODE multiply repeatedly by \( x \) until this is no longer the case.