2. INTEGRATION

§2.1. A Review of Integral Calculus

An (indefinite) integral of a function \( f(x) \), where one exists, is a function \( I(x) \) whose derivative is \( f(x) \). A function is defined to be integrable if it has an integral. Any two integrals differ by a constant and we write them as \( \int f(x)\,dx = g(x) + c \), where \( c \) is called an arbitrary constant.

If \( u, v \) are functions of \( x \) and \( c \) is a constant then
\[
\int (u + v)\,dx = \int u\,dx + \int v\,dx \quad \text{and} \quad \int cu\,dx = c\int u\,dx.
\]

Polynomials are integrable and their integrals can be obtained from the fact that:
\[
\int x^n\,dx = \begin{cases} 
\frac{1}{n+1}x^{n+1} & \text{if } n \neq -1 \\
\log|x| & \text{if } n = -1
\end{cases}.
\]

The integral of \( e^x \) is \( e^x \) and the integral of \( \sin x \) is \( -\cos x \).

If \( a \leq b \) and \( I(x) = \int_a^b f(x)\,dx \) then \( \int_a^b f(x)\,dx \) denotes \( I(b) - I(a) \), usually written as \( [I(x)]_a^b \).

This is called a definite integral and if \( f(x) \geq 0 \) between \( x = a \) and \( x = b \) it gives the area between the graph \( y = f(x) \) and the x-axis between \( x = a \) and \( x = b \).

§2.2. The Riemann Integral

We have defined integrals as anti-derivatives. This is fine, up to a point, but it means that we can only prove that an integral exists if we can actually find it. Most integrals cannot be expressed in terms of elementary functions, such as polynomials, trigonometric, logarithmic or exponential functions. We have to define new functions as integrals.

For example we might define a function \( \phi(x) = \int e^{\frac{1}{2}x^2}\,dx \). But how do you make a definition in terms of something until you know what that something is? To get around this difficulty we define an integral in a way that is independent of knowing what the integral is. There are several integrals that are used. They all give the same answer whenever they exist. It’s just that some integrals can be defined wherever others do not. We will here deal with the simplest of these integrals, the Riemann integral.
Suppose \( f(x) \) is continuous in the closed interval \([a, b]\). We define a **partition** of \([a, b]\) to be a finite sequence \( \Delta = (x_0, x_1, ..., x_n) \) such that
\[
a = x_0 < x_1 < ... < x_n = b.
\]
We define the **resolution** of the partition \( \Delta \) to be the maximum of \( \{x_{i+1} - x_i\} \) and we denote it by \(|\Delta|\).

A **subpartition** of \( \Delta \) is a sequence \((x_0, t_0, x_1, t_1, x_2, ..., x_n)\) where, for each \( i \), \( x_i \leq t_i \leq x_{i+1} \). The **Riemann sum** corresponding to this partition and subpartition is
\[
\sum_{i=0}^{n-1} (x_{i+1} - x_i) f(t_i).
\]

Two special cases are the following. Suppose \( m_i \) is the minimum value of \( f(x) \) on the interval \([x_i, x_{i+1}]\) and suppose \( M_i \) is the maximum value of \( f(x) \) on this same interval. The **lower Riemann sum** of \( f(x) \) that corresponds to the partition \( \Delta \) is defined to be
\[
s_\Delta(f) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f(m_i)
\]
and the **upper Riemann sum** is defined to be
\[
S_\Delta(f) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) f(M_i).
\]
Clearly every Riemann sum of \( f(x) \), for the partition \( \Delta \) lies between these two. That is
\[
s_\Delta(f) \leq \sum_{i=0}^{n-1} (x_{i+1} - x_i) f(t_i) \leq S_\Delta(f).
\]

Now what exactly is going on here? We are leading up to a definition of the integral as the (signed) area under the curve. For convenience in this explanation suppose that \( f(x) \geq 0 \) for \( x \in [a, b] \), though everything works without this assumption.

Suppose we draw rectangles up from the \( x \)-axis using ordinates at the \( x_i \) and suppose that the heights are the values of \( f(t_i) \).

Note that we do not insist on a dissection with equal widths. The area of the rectangles will be approximately the area under the curve. In fact the area under the curve will lie between the lower Riemann sum and the upper one.

The sum of the areas in the following diagram is the lower Riemann sum.
The sum of the areas in the following diagram is the upper Riemann sum.

If the limits of \( s_\Delta(f) \) and \( S_\Delta(f) \) as \( |\Delta| \) approaches 0 exist and are equal we define this value to be the \textbf{integral} of \( f(x) \) over the interval \([a, b]\). We write this value as \( \int_a^b f(x) \, dx \). We say that \( f(x) \) is \textbf{integrable} over \([a, b]\) if such an integral exists and it is \textbf{integrable} if it is integrable over every closed interval.

Example 1: The function \( f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 4 - 2x & \text{if } x > 1 \end{cases} \) is integrable on \([0, 2]\).

Solution:

Take the partition \( \Delta_{m,n} \): \( 0 < \frac{1}{n} < \frac{2}{n} < \ldots < 1 < 1 + \frac{1}{m} < 1 + \frac{2}{m} < \ldots < 2 \), for some positive integers \( m, n \). (Here we are dividing the interval \([0, 1]\) into \( m \) equal parts and the interval \([1, 2]\) into \( m \) equal parts. There is no good reason for different widths for each interval, except to emphasise that we do not need to have a uniform partition.)

Here \( |\Delta_{m,n}| = \max \left( \frac{1}{m} - \frac{1}{n} \right) \).

Recall that \( \sum_{r=1}^n = \frac{1}{2} n(n+1) \) and \( \sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1) \).
The lower Riemann sum for this partition is
\[ \sum_{r=0}^{n-1} \left( \frac{1}{n} r^2 + \frac{1}{m} \left( 2 - \frac{2(r + 1)}{m} \right) \right) \]
\[ = \frac{1}{n^3} \sum_{r=0}^{n-1} r^2 + \frac{2m}{m^2} \sum_{r=0}^{m-1} (r + 1) \]
\[ = \frac{(n-1)n(2n-1)}{6n^3} + 2 - \left( \frac{2}{m^2} \right) \left( \frac{m^2 + m}{2} \right) \]
\[ = \frac{1}{3} - \frac{3n-1}{6n^2} + 1 - \frac{1}{m} \]
\[ = \frac{4}{3} - \frac{3n-1}{6n^2} - \frac{1}{m} \]

The upper Riemann sum for this partition is
\[ \sum_{r=0}^{n-1} \left( \frac{1}{n} (r + 1)^2 + \frac{1}{m} \left( 2 - \frac{2r}{m} \right) \right) \]
\[ = \frac{1}{n^3} \sum_{r=0}^{n-1} (r + 1)^2 + \frac{2m}{m^2} \sum_{r=0}^{m-1} r \]
\[ = \frac{n(n + 1)(2n + 1)}{6n^3} + 2 - \left( \frac{2}{m^2} \right) \left( \frac{m^2 - m}{2} \right) \]
\[ = \frac{1}{3} + \frac{3n+1}{6n^2} + 1 + \frac{1}{m} \]
\[ = \frac{4}{3} + \frac{3n+1}{6n^2} + \frac{1}{m} \]

As \( |\Delta_{m,n}| \to 0 \) both of these approach \( \frac{4}{3} \) so we conclude that \( \int_0^2 f(x) \, dx = \frac{4}{3} \).

Before we knew about the Riemann integral we would have written
\[ \int_0^2 f(x) \, dx = \int_0^1 x^2 \, dx + \int_1^2 (4 - 2x) \, dx \]
\[ = \left[ \frac{x^3}{3} \right]_0^1 + [4x - x^2]_1^2 \]
\[ = \frac{1}{3} + (4 - 3) = \frac{4}{3} \]

That is because we thought of integrals as antiderivatives. And once we prove the
Fundamental Theorem of Calculus we can drop back into our old habits. So why all the fuss
about the Riemann integral? The answer is that there are integrals, such as \( \int e^{-x^2} \, dx \) where we cannot find an existing function whose derivative is \( e^{-x^2} \). So we might define a new function \( \phi(x) = \int_0^x e^{-t^2} \, dt \). But this involves circular reasoning. We need to define the integral independently of differentiation, and this is what the Riemann integral does.

Now our example shows that a function can be integrable even if it has a discontinuity or two. Too many discontinuities, however, may render the function non integrable.

**Example 2:** The function \( f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \) is not integrable on \([0, 1]\).

**Solution:** For any partition the lower Riemann sum will be 0 and the upper Riemann sum will be one. Hence, although the limits of these Riemann sums exist they are not equal.

The following theorem guarantees the integrability of most of the functions we find useful. The proof can be found in books on Real Analysis.

**Theorem 1:** If \( f(x) \) is continuous on the interval \([a, b]\) then it is integrable on \([a, b]\).

**Example 3:** The function \( f(x) = e^{-x^2} \) is integrable because it is continuous. The fact that we cannot express the integral in terms of the elementary functions does not affect its integrability.

We really should have used the term “Riemann integrable” instead of “integrable” because there are generalisations of the Riemann integral, such as the Lesbegue integral which can provide integrals for some functions for which the Riemann integral does not exist. We shall not discuss this here as it properly belongs to an advanced area of mathematics called “Measure Theory”.

At this stage we could prove all the familiar properties of the integral directly from the Riemann integral definition. However we do not think it very instructive to do so. You will have seen proofs of these using the properties of the derivative, based on the fact that integration is anti-differentiation. This fact is enshrined in the so-called Fundamental Theorem of Calculus.
Theorem 2: (Fundamental Theorem of Calculus): Suppose \( f(x) \) is a continuous function on the interval \([a, b]\) and suppose that \( F(x) \) is the Riemann integral of \( f(x) \) on the interval \([a, x]\), where \( a \leq x \leq b \). Then \( F(x) \) is differentiable on \([a, b]\) and \( \frac{dF(x)}{dx} = f(x) \). Also, if \( f(x) \) is differentiable on \([a, b]\) then \( f'(x) \) is integrable and \( \int_a^x f'(t) \, dt = f(x) - f(a) \).