§10.1. Systems of Polynomials in Several Variables

An expression such as \( x^3y^2 + 3x^2y + y^2 - 4x + 7 \) is a polynomial in two variables \( x, y \). We denote the set of all such polynomials over the field of coefficients \( F \), by \( F[x][y] \). Many of the one-variable ideas carry across to two or more variables.

A polynomial in \( x, y \) can be regarded as a polynomial in \( x \) whose coefficients are polynomials in \( y \), or vice versa. For example, we could consider the above polynomial as being the cubic in \( x \):

\[
y^2x^3 + 3yx^2 - 4x + (y^2 - 7),
\]

or as a quadratic in \( y \):

\[
(x^3 + 1)y^2 + 3x^2y + (-4x + 7).
\]

We could try to copy the one variable methods to solve a system of equations in two or more variables. But it only works up to a point. If we treat \( f(x, y) \) and \( g(x, y) \) as polynomials in \( y \) with the coefficients being polynomials in \( x \) we can calculate their GCD only if we allow our calculations to involve not just polynomials in \( x \), but rational functions — fractions with polynomial numerators and denominators.

Fine, we can do that, and we can even prove that the GCD can be expressed in the form \( a(x)f(x,y) + b(x)g(x,y) \). What breaks down is that solving the GCD need not be equivalent to solving the system. This is because the \( a(x) \) and \( b(x) \) will not, in general, be polynomials. They will be rational functions. But won't it still be true that if \( f(x, y) \) and \( g(x, y) \) are simultaneously zero then the this linear combination must be as well?

Well, consider the rather trivial system:

\[
\begin{cases}
x = 0 \\
y = 0
\end{cases}
\]

The GCD of \( x \) and \( y \) is clearly 1, and indeed 1 can be expressed in the required form:

\[
1 = \frac{1}{x^2}x + 0.y.
\]

But putting \( x = 0, y = 0 \) doesn't give us \( 1 = 0 \). The situation with polynomials in two or more variables is rather more complicated.

We can, however, go part of the way. We can multiply an equation by a polynomial and add or subtract equations, but we must not divide. Adding or subtracting equations will result in an equivalent system. Multiplying can introduce “false” solutions, so we must check that no solution we found was introduced when we multiplied.

**Example 1:** Solve the system:

\[
\begin{cases}
x^2y - 3xy^3 + 3y = 1 \\
x^2y - 3xy^3 + 2y = 0
\end{cases}
\]

**Solution:** Subtracting the second equation from the first we get \( y = 1 \). Substituting into the second equation we get \( x^2 - 3x + 2 = 0 \) which has solutions \( x = 1 \) and \( x = 2 \). So the system has two solutions for \((x, y)\), viz. \((1, 1)\) and \((2, 1)\).
Example 2: Find the real solutions to the system:
\[
\begin{align*}
    xy^2 + x^3 + 16 &= 0 \\
    x^2y^2 + x^5 + 16 &= 0
\end{align*}
\]
Solution: Subtracting x times the first equation from the second we get \(x^5 - x^4 + 16 - 16x\). Thus \((x^4 - 16)(x - 1) = 0\). This gives real solutions \(x = 1, \pm 2\). Substituting into the first equation gives the corresponding values of \(y\). The solutions \((x, y)\) are thus: \((1, \sqrt[3]{15}), (1, -\sqrt[3]{15}), (2, 2), (2, -2)\) N.B. There are no real solutions for \(y\) if \(x = -2\).

NOTE: A system of linear equations must either have no solutions, a single solution, or infinitely many solutions. A system of polynomial equations can have any number of solutions, finite or infinite.

There are systematic algorithms for systems of polynomial equations, analogous to the Gaussian reduction to echelon form for linear systems. The method is known as finding a Gröbner Basis for the system. A considerable amount of work has been done in the last decade on Gröbner Bases in the context of Computational Algebra. However it is too technical for us to consider here. We shall content ourselves with simple-minded ad hoc methods as in the above examples.

Curves in can be defined by a polynomial equation in two or more variables and their points of intersection correspond to the solutions of systems of such polynomials. Such curves, defined implicitly by means of polynomial equations, are known as algebraic curves and the interplay of the polynomial algebra with the geometry is studied in the branch of mathematics known as Algebraic Geometry.

Example 3: Solve the system:
\[
\begin{align*}
    x^3 + y^3 &= 279 \\
    x^2 + y^2 &= 65
\end{align*}
\]
Solution: Subtract \(y\) times the second equation from the first. This produces:
\[
\begin{align*}
    x^3 - x^2y &= 279 - 65y \\
    y(x^2 - 65) &= x^3 - 279.
\end{align*}
\]
Squaring gives \(y^2(x^2 - 65)^2 = (x^3 - 279)^2\) and substituting \(y^2 = 65 - x^2\), we get \((x^3 - 279)^2 + (x^2 - 65)^4 = 0\). This gives us a polynomial of degree 8 in \(x\) which we can in principle solve by graphing and homing in on the solutions by Newton's Method. Then for each such \(x\) we get \(y = \pm\sqrt{65 - x^2}\). O.K. in principle, but ugh! Who wants to go to all that trouble. There has to be an easier way. Trial and error? Clearly \(x = 1, y = 8\) works for the second equation but not the first.

Actually there is a simple, integer solution for \(x\) and \(y\) that you can find without too much trouble. But trial and error won't guarantee to pick up all the solutions.
§10.2. Exploiting Symmetry

The system of equations in Example 3 are both symmetric in x and y. If (x, y) is a solution then so is (y, x). We can exploit this symmetry to simplify our working.

If we have the system:

\[
\begin{align*}
  f(x, y) &= 0 \\
  g(x, y) &= 0
\end{align*}
\]

and f and g are symmetric in x and y, we can express them in terms of x + y and xy.

Then put S = x + y and P = xy to give us a new system in S and P. Perhaps this new system will be easier to solve.

Example 3 (again): Solve the system:

\[
\begin{align*}
  x^3 + y^3 &= 279 \\
  x^2 + y^2 &= 65
\end{align*}
\]

Solution: Let S = x + y and P = xy. Then \(x^3 + y^3 = (x + y)^3 - 3x^2y - 3xy^2 = S^3 - 3PS\) and \(x^2 + y^2 = S^2 - 2P\). This gives us a new system:

\[
\begin{align*}
  S^3 - 3PS &= 279 \\
  S^2 - 2P &= 65
\end{align*}
\]

What have we gained by doing this? In terms of S we have a quadratic and a cubic, as before. But in terms of P we have two linear polynomials. So \(P = \frac{S^3 - 279}{3S} = \frac{S^2 - 65}{2}\) (dividing by S is O.K. because S = 0 is not possible — why not?).

Thus \(2S^3 - 558 = 3S^3 - 195S\) and so \(S^3 - 195S + 558 = 0\). There are three real solutions to this cubic, \(S = 3\) and approximately \(S = -15.22\) and 12.22. It is now easy to find the corresponding values of P. The solutions for S and P are (approximately):

\[
\begin{align*}
  S &= -15.22, P = 83.32; \\
  S &= 3, P = -28; \\
  S &= 12.22, P = 42.16
\end{align*}
\]

Now, in each of these cases, we must solve the system:

\[
\begin{align*}
  x + y &= S \\
  xy &= P
\end{align*}
\]

The values of x, y will simply be the roots of the quadratic equations \(x^2 - Sx + P = 0\).

Solving these three quadratics we get six solutions for x and y, approximately:

\(-4, 7), (7, -4), (-7.61 +10.08 i, -7.61 -10.08 i), (-7.61 -10.08 i, -7.61 +10.08 i), (6.11 + 4.39 i, 6.11 - 4.39 i), (6.11 - 4.39 i, 6.11 + 4.39 i)\)
As noted earlier, we can solve a polynomial equation \( f(z) \) in a complex variable \( z \) by putting \( z = x + iy \) and equating real and imaginary parts. This gives a system of two polynomial equations in \( x \) and \( y \) which we proceed to solve. Of course in this situation we are only interested in real solutions for \( x \) and \( y \).

**Example 4:** Find all solutions to the equation \( z^3 + z + 1 = 0 \).

**Solution:** Let \( z = x + iy \) and equate real and imaginary parts to obtain the system:

\[
\begin{align*}
  x^3 - 3xy^2 + x + 1 &= 0 \\
  3x^2y - y^3 + y &= 0
\end{align*}
\]

From the second equation we get \( y = 0 \) or \( 3x^2 - y^2 + 1 = 0 \). Substituting \( y = 0 \) into the first equation gives us \( x^3 + x + 1 = 0 \). This is simply reminding us that we can expect a real solution. We can use Newton's Method to find it. It is approximately \( x = -0.682 \ldots \)

Suppose now that \( y \neq 0 \). We thus have the system:

\[
\begin{align*}
  x^3 - 3xy^2 + x + 1 &= 0 \\
  3x^2 - y^2 + 1 &= 0
\end{align*}
\]

Although these are both quadratic in \( y \), neither has a term in \( y \). So in fact they are both linear in \( y^2 \). We can thus obtain:

\[
y^2 = \frac{x^3 + x + 1}{3x} = 3x^2 + 1 \text{ and so obtain the cubic equation } 2x^3 + 2x - 1 = 0.
\]

Since we are only interested in the real roots of this cubic we can locate the single real root near 0.4. We can use Newton's Method if we need to find it more accurately 0.424...

Using \( y^2 = 3x^2 + 1 \), we find that there are two corresponding values of \( y \): \( \pm 1.24i \).

The three solutions to the cubic are thus approximately \(-0.682, 0.424 \pm 1.24i \).
EXERCISES FOR CHAPTER 10

Exercise 1: Solve the system:
\[
\begin{align*}
xy^3 + x^2y^2 + 18 &= 0 \quad (1) \\
x^2y^2 + x^3y - 12 &= 0 \quad (2)
\end{align*}
\]

Exercise 2: Solve the system:
\[
\begin{align*}
x^3y + xy^3 &= 14 \\
x^2 + y^2 - xy &= 5
\end{align*}
\]

Exercise 3: Solve the cubic \( z^3 - 2z - 8 = 0 \).

Exercise 4: Solve the cubic equation:
\( z^3 - 7z - 6 = 0 \).

Exercise 5: Solve the system:
\[
\begin{align*}
x^6 + x^5 + x^4 + x^2 + x + 1 &= 0 \\
x^5 + x^4 + 2x^3 + 2x^2 + 2x + 1 &= 0
\end{align*}
\]

Exercise 6: Solve the system:
\[
\begin{align*}
x^2 + y^2 + x + y &= 12 \\
x^2y + xy^2 + 2x + 2y &= -2
\end{align*}
\]

Exercise 7: Solve the system:
\[
\begin{align*}
x^5y + x^3y^3 + x^2y^4 - 2 &= 0 \\
x^6 + x^4y^2 + x^3y^3 + 1 &= 0
\end{align*}
\]

Exercise 8: Solve \( z^3 + z + i = 0 \).

Exercise 9: Prove that if \( z = x + iy \) is a non-real root of a monic real cubic \( f(z) \), then \( x \) is a real root of the cubic \( 2f(x) - f'(x)f''(x) \).

SOLUTIONS FOR CHAPTER 10

Exercise 1: Multiplying (1) by \( x \) and (2) by \( y \) (to get rid of the \( y^3 \)) we obtain:
\[
\begin{align*}
x^2y^3 + x^3y^2 + 18x &= 0 \quad (3) \\
x^2y^2 + x^3y^2 - 12y &= 0 \quad (4)
\end{align*}
\]
Subtracting, we get \( 18x + 12y = 0 \) and so \( y = -3x/2 \). Now substituting into (2) we get:
\[
x^2 \left( 9x^2 \right) + x^3 \left( \frac{-3x}{2} \right) = 12 \text{ and hence } 3x^4 = 48 \text{ i.e. } x^4 = 16.
\]
So the solutions for \( x \) are \( 2, -2, 2i, -2i \) and hence the four solutions for \( (x, y) \) are:
\( (2, -3), (-2, 3), (2i, -3i), (-2i, 3i) \).
**Exercise 2:** This system is symmetric in \(x\) and \(y\). Therefore put \(S = x + y\) and \(P = xy\).
The system reduces to:

\[
\begin{align*}
\text{P}(S^2 - 2P) &= 14 \\
S^2 - 3P &= 5
\end{align*}
\]
Noting the similarity of the \(S^2 - 2P\) with the \(S^2 - 3P\) we can write the first equation as:

\[
P(S + 5) = 14.
\]
Hence \(P^2 + 5P - 14 = (P - 2)(P + 7) = 0\) and so \(P = 2\) or \(-7\).
If \(P = 2\), \(S^2 = 3P + 5 = 11\) and so \(S = \pm \sqrt{11}\).
This leads to the two quadratics in \(x\), \(y\):

\[
x^2 \pm \sqrt{11} x + 2 \text{ giving the solutions:}
\]

\[
x = \frac{-\sqrt{11} + \sqrt{3}}{2}, \ y = \frac{-\sqrt{11} - \sqrt{3}}{2} \quad \text{(or vice versa)}
\]

and

\[
x = \frac{-\sqrt{11} + \sqrt{3}}{2}, \ y = \frac{-\sqrt{11} - \sqrt{3}}{2} \quad \text{(or vice versa)}.
\]

If \(P = -7\), \(S = \pm 4i\) giving the quadratics:

\[
x^2 \pm 4i x - 7 \text{ and hence the solutions:}
\]

\[
x = \sqrt{3} + 2i, \ y = -\sqrt{3} + 2i \quad \text{(or vice versa)} \quad \text{and} \quad x = \sqrt{3} - 2i, \ y = -\sqrt{3} - 2i \quad \text{(or vice versa)}.
\]

**Exercise 3:** Put \(z = x + iy\). Then

\[(x+iy)^3 - 2(x+iy) - 8 = 0.\]
Equating real and imaginary parts we arrive at the system:

\[
\begin{align*}
x^3 - 3xy^2 - 2x - 8 &= 0 \\
3x^2y - y^3 - 2y &= 0
\end{align*}
\]
If \(y \neq 0\) the last equation reduces to:

\[3x^2 - y^2 - 2 = 0 \text{ and so } y^2 = 3x^2 - 2.\]
Substituting into the first equation we get:

\[x^3 - 3x(3x^2 - 2) - 2x - 8 = 0.\]
Simplifying:

\[8x^3 - 4x + 8 = 0 \text{ and so } f(x) = 2x^3 - x + 2 = 0.\]
This has a real root close to \(x = -1\).
Now \(f'(x) = 6x^2 - 1\) and by Newton’s Method we get a a better approximation:

\[-1 - f(-1)/f'(-1) = -1 - 1/5 = -1.2\]
An even better approximation is:

\[-1.2 - f(-1.2)/f'(-1.2) = -1.2 + \frac{0.2567}{7.64} = -1.166.\]
The corresponding values of \(y\) are \(\pm \sqrt{3(-1.166)^2 - 2} = \pm \sqrt{2.079} = \pm 1.442\).
There are thus two non-real roots: \(-1.166 \pm 1.442 i\).

Since the sum of the roots is 0, the real root must be approximately 4.332.

**Exercise 4:** Putting \(z = x + iy\) as above we get the system:

\[
\begin{align*}
  x^3 - 3xy^2 - 7x - 6 &= 0 \\
  3x^2y - y^3 - 7y &= 0 \\
\end{align*}
\]

Dividing the second equation by \( y \) we get \( y^2 = 3x^2 - 7 \).
Substituting into the first equation, and simplifying, we get \( 4x^3 - 7x + 3 = 0 \).
Clearly \( x = 1 \) is one root and factorising we get \( 4x^3 - 7x + 3 = (x - 1)(4x^2 + 4x - 3) = (x - 1)(2x - 1)(2x + 3) \).
So \( x = 1, 1/2 \) and \(-3/2\). The values of \( y^2 \), from \( y^2 = 3x^2 - 7 \) are \(-4, -6.25, -4.75\).

But wait, there appears to be something wrong here! The values of \( x \) and \( y \) have to be real so \( y^2 \) can’t be negative.

Does this mean that there are no roots? No. Every cubic has three roots. Moreover, one of them is real.

What has happened is that in dividing by \( y \) we were implicitly assuming that \( y \neq 0 \), that is, we were only going in search of non-real roots. As it was we showed that there were none. All three roots must be real. This can be confirmed by sketching the original cubic. In fact it has roots at \( z = -1, -2 \) and 3.

**Exercise 5:**
The GCD of these two polynomials can be shown to be \( x^2 + x + 1 \). Hence the common solutions to these two equations are \( \omega \) and \( \omega^2 \).
Exercise 6:
This system is symmetric in x, y. Let S = x + y and P = xy.
Then $S^2 - 2P + S = 12$ and $PS + 2S = -2$.
From the first equation $P = \frac{1}{2} (S^2 + S - 12)$.
Substituting into the second we get $S^3 + S^2 - 8S + 4 = 0$.
By inspection, $S = 2$ is a solution. Hence $S - 2$ is a factor of this cubic.
The other factor can be easily found to be $S^2 + 3S - 2$, giving $S = \frac{-3 \pm \sqrt{17}}{2}$.
The corresponding values of P are $-3, -\frac{7}{2}, -\frac{7}{2}$.
The solutions x, y are the two zeros of the three quadratics $x^2 - Sx + P$.
Thus $x, y = S \pm \sqrt{S^2 - 4P}$.
If $S = 2$, $P = -3$, $x, y = 2.17256, -1.61100$.
If $S = -\frac{3 + \sqrt{17}}{2}$, $y \approx 1.3247$. Hence $x \approx -0.5624$.
Hence the three solutions to $z^3 + z + i = 0$ are, approximately:
$1.3247i$, $0.5624 - 0.6624i$ and $-0.5624 - 0.6624i$.

Exercise 7:
(Equation 1) $x - (Equation 2 \times y)$ is $-2x - y = 0$, so $y = -2x$.
Substituting into Equation 2 we get $x^6 + 4x^6 - 8x^6 + 1 = 0$, that is, $x^6 = \frac{1}{3}$.
Hence $x \approx 0.8327, y \approx -1.6654$ or $x \approx -0.8327, y \approx 1.6654$.

Exercise 8: Let $z = x + iy$.
Then $(x + iy)^3 + (x + iy) + i = 0$.
Equating real and imaginary parts we get the system:
\[
\begin{align*}
x^3 - 3xy^2 + x &= 0 \\
3x^2y - y^3 + y + 1 &= 0
\end{align*}
\]
If $x = 0$ then $y^3 - y - 1 = 0$.
By Newton’s Method we get $y \approx 1.3247$.
Now suppose that $x \neq 0$.
Then, from the first equation, $x^2 - 3y^2 + 1 = 0$, and so $x^2 = 3y^2 - 1$.
Substituting into the second equation we get $3(3y^2 - 1)y - y^3 + y + 1 = 0$, that is $8y^3 - 2y + 1 = 0$.
By Newton’s Method we get $y \approx -0.6624$. Hence $x^2 = 0.3163$ and so $x = \pm 0.5624$.
Hence the three solutions to $z^3 + z + i = 0$ are, approximately:
$1.3247i$, $0.5624 - 0.6624i$ and $-0.5624 - 0.6624i$.
**Exercise 9:** Let \( f(z) = z^3 + az^2 + bz + c \), where \( a, b, c \) are real, and let \( z = x + iy \).

Then \((x + iy)^3 + a(x + iy)^2 + b(x + iy) + c = 0\).

Equating real and imaginary parts we get the system:
\[
\begin{align*}
\begin{cases}
\begin{align*}
x^3 - 3xy^2 + ax^2 - ay^2 + bx + c &= 0, \\
3x^2y - y^3 + 2axy + by &= 0
\end{align*}
\end{cases}
\]
\]

Since \( z \) is non-real, \( y \neq 0 \) and so \(3x^2 - y^2 + 2ax + b = 0\).

So \( y^2 = 3x^2 + 2ax + b\).

Substituting into the first equation we get
\[
(x^3 + ax^2 + bx + c) - (3x + a)(3x^2 + 2ax + b) = 0
\]

and so
\[
2(x^3 + ax^2 + bx + c) - (6x + 2a)(3x^2 + 2ax + b) = 0.
\]

In other words,
\[
2f(x) - f''(x)f'(x) = 0.
\]