10. ORDINAL NUMBERS

§10.1 Transitive Sets

We turn our attention to ordinal numbers. With finite numbers the cardinal numbers, apart from zero, are 1, 2, 3, … while the ordinal numbers are 1st, 2nd, 3rd, … There really is very little difference. For infinite sets there is a big difference. While cardinal numbers simply measure the size of a set, ordinal numbers describe the structure of a well-ordered set.

Consider the sequence 1, 2, 3, … , ∞ where “∞” is a symbol that is defined to be greater than every finite number. As a set this is no bigger than the set {1, 2, 3, …}. But the ordering is quite different. One set has a largest while the other does not.

A set x is transitive if ∪x ⊆ x, that is if every element of an element of x is itself an element of x.

Example 1: \{0, 1, 2, \{2\}, \{1, 2\}, (2, 1)\} is transitive.

The natural numbers are transitive, as is the set of natural numbers.

Theorem 1: A set x is transitive if and only if x ⊆ øx.

Suppose x is trans. Let y ∈ x and let
z ∈ y. Then z ∈ ∪x and so z ∈ x. Hence y ⊆ x. Suppose x ⊆ øx. Let
y ∈ ∪x. Then y ∈ z for some z ∈ x ⊆ øx. So z ⊆ x and hence y ∈ x.

Theorem 2: If the elements of x are transitive then so is ∩x.

Proof: Let y ∈ z ∈ ∩x. Let u ∈ x. Then y ∈ z ∈ u and since u is transitive, y ∈ u. Thus
y ∈ ∩x Hence ∩x is transitive.

Theorem 3: If the elements of x are transitive then so is ∪x.

Proof: Let y ∈ z ∈ ∪x. Then z ∈ u for some u ∈ x. Then y ∈ z ∈ u and since u is transitive, y ∈ u. Thus y ∈ ∪x and so ∪x is transitive.

§10.2 Ordinal Numbers

An ordinal number is a transitive set which is well-ordered by the relation “∈ or =”. So, for ordinals, < and ∈ are equivalent. We denote the class of ordinals by Ord.

Examples 2: Ordinal numbers include all the natural numbers, as well as ω and ω+.

If a well-ordered set X is similar to the ordinal α we say that α is its ordinal number. The above property shows that this is uniquely defined. We denote the ordinal of X by ord (X, ≤). If the ordering is understood we can write ord(X).

Theorem 3: If α is an ordinal number, then so is α+.
**Theorem 4:** Elements of ordinals are ordinals.

**Proof:** Elements of ordinals are subsets and so are well-ordered. Let \( x \in y \in z \in \alpha \), where \( \alpha \) is an ordinal. Then since \( \alpha \) is transitive, \( x, y, z \) are elements of \( \alpha \). Since \( \in \) is transitive on \( \alpha \), \( x \in z \).

**Theorem 5:** Similar ordinals are equal.

**Proof:** Let \( f: \alpha \to \beta \) be a similarity. Suppose \( x \) is the smallest element of \( \alpha \) such that \( f(x) \neq x \). If \( y < x \) then \( y = f(y) < f(x) \), whence \( x \subseteq f(x) \). Now suppose \( y < f(x) \). Then, since \( f^{-1} \) is a similarity, \( f^{-1}(y) < x \). Hence \( y = f(f^{-1}(y)) = f f^{-1}(y) < x \). So \( f(x) \subseteq x \), a contradiction. Hence there is no such \( x \).

**Theorem 6:** If \( \alpha \) and \( \beta \) are ordinal numbers and \( \alpha \subseteq \beta \) then \( \alpha \) is an element of \( \beta \).

**Proof:** Suppose \( \alpha, \beta \) are ordinals such that \( \alpha \subseteq \beta \). Let \( x \) be the least element of \( \beta \). Since \( \beta \) is transitive, \( x \subseteq \beta \). Since \( \{x, y\} \) has a least, either \( x \in y \) or \( x = y \) or \( y \in x \). If \( x \in y \) or \( x = y \), then \( x \in \alpha \), contradicting the fact that \( x \subseteq \beta \). Thus \( y \in x \) and so \( \alpha \subseteq x \). But \( x \subseteq \alpha \). Hence \( x = \alpha \). As \( x \in \beta \), \( \alpha \in \beta \).

**Theorem 7:** Transitive subsets of ordinals are ordinals.

**Proof:** Suppose there is an ordinal \( \alpha \) having a transitive subset which is not itself an ordinal, and suppose \( \beta \) is the least element of \( \beta - \alpha \). Since \( \beta - \alpha \) is transitive, \( x \subseteq \beta \). Since \( (\beta - \alpha) \cap x = 0 \), \( x \subseteq \alpha \). Let \( y \in \alpha \). Then \( y \in \beta \). Since \( \{x, y\} \) has a least, either \( x \in y \) or \( x = y \) or \( y \in x \). If \( x \in y \) or \( x = y \), then \( x \in \alpha \), contradicting the fact that \( x \subseteq \beta - \alpha \). Thus \( y \in x \) and so \( \alpha \subseteq x \). But \( x \subseteq \alpha \). Hence \( x = \alpha \). As \( x \in \beta \), \( \alpha \in \beta \).

**Theorem 8:** If \( X \) is a set of ordinals then \( \bigcup X \) is an ordinal.

**Proof:** Suppose \( X \) is a set of ordinals. Then \( \bigcup X \) is transitive. Let \( 0 \neq Y \subseteq \bigcup X \). Then the elements of \( Y \) are ordinals and so \( \cap Y \) is transitive. Let \( \alpha \in Y \). Then \( \cap Y \subseteq \alpha \). Thus \( \cap Y \) is an ordinal. Hence \( \cap Y \in \alpha \) or \( \cap Y = \alpha \in Y \). Hence if \( \cap Y \notin Y \), \( \cap Y \in \cap Y \), a contradiction. Thus \( \cap Y \) is the least element of \( Y \). Hence \( \bigcup X \) is well-ordered by \( \in \). Thus \( \bigcup X \) is an ordinal.

**Theorem 9 (Burali-Forti Paradox):** The class of ordinals is not a set.

**Proof:** Suppose \( \text{Ord} \) is a set. Then \( \bigcup \text{Ord} \in \text{Ord} \) and so \( \bigcup \text{Ord} \in (\bigcup \text{Ord}) \) \( \in \text{Ord} \) so \( \bigcup \text{Ord} \in \bigcup \text{Ord} \), a contradiction.
§10.3 Transfinite Induction

The method of Proofs by Induction is very useful in mathematics. It works because every non-empty set of natural numbers has a least, that is, the set of natural numbers is well-ordered by the usual ordering. Finite induction can be extended to infinite sets, provided we can well-order them.

**Theorem 10 (Proof by Transfinite Induction):** Suppose $W$ is a well-ordered set and $X$ is a subset of $W$. Suppose that $X$ has the property that whenever all the predecessors of $x$ are in $X$ then so is $x$. Then $X = W$.

**Proof:** Suppose $X \subseteq W$. Then $m = \min(W - X) \rightarrow C!$

We can also define things by transfinite induction. The following is a special case.

**Theorem 11:** Suppose $G$ is a generalized function whose domain is $S$, a subset of Ord with no maximum. Then there exists a unique function $f$ on $S$ such that $f(x^+) = G(f(x))$ for all $x \in S$ and $f(x) = \cup \{f(y) \mid y < x\}$ if $x$ has no predecessor.

§10.4 The Arithmetic of Ordinal Numbers

A non-zero ordinal is a **limit ordinal** if it has no immediate predecessor. An obvious example is $\omega$.

We define **addition** of ordinals by transfinite induction as follows:

- (A0) $\alpha + 0 = \alpha$;
- (A1) $\alpha + \beta^+ = (\alpha + \beta)^+$;
- (A2) $\alpha + \beta = \cup \{\alpha + \gamma \mid \gamma < \beta\}$ if $\beta$ is a limit ordinal.

**Example 3:** Successively adding 1 to $\omega$ we get the sequence $\omega + 1, \omega + 2, \ldots$

We define **multiplication** of ordinals by transfinite induction as follows:

- (M0) $\alpha 0 = 0$;
- (M1) $\alpha \beta^+ = (\alpha \beta) + \alpha$;
- (M2) $\alpha \beta = \cup \{\alpha \gamma \mid \gamma < \beta\}$ if $\beta$ is a limit ordinal.

We define **exponentiation** of ordinals by transfinite induction as follows:

- (E0) $\alpha^0 = 1$;
- (E1) $\alpha \beta^+ = \alpha\beta.\alpha$;
- (E2) $\alpha^\beta = \cup \{\alpha + \gamma \mid \gamma < \beta\}$ if $\beta$ is a limit ordinal and $\alpha \neq 0$;
- (E3) $0^\beta = 0$ if $\beta$ is a limit ordinal.

**Examples 4:**

- $\omega + 1 = \omega + 0^+ = (\omega + 0)^+ \text{ by (A1)} = \omega^+ \text{ by (A0)} \neq \omega$.
- $1 + \omega = \cup \{1 + n \mid n < \omega\} = \omega$.
- $2\omega = \cup \{2n \mid n < \omega\} \text{ by (M2)} = \omega$.
- $\omega2 = \omega1^+ = \omega1 + \omega \text{ by (M1)} = \omega0^+ + \omega = (\omega0 + \omega) + \omega \text{ by (M1)}$
- $= (0 + \omega) + \omega \text{ by (M0)} = \cup \{0 + n \mid n < \omega\} + \omega = \omega + \omega = \cup \{\omega + n \mid n < \omega\} \text{ by (A2)}$.  

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\[ 2^\omega = \bigcup \{ 2^n \mid n < \omega \} \] by (E2) = \omega

§10.5 Further Arithmetic of Cardinal Numbers

**Theorem 12:** If a, b are cardinal numbers and a is finite and b is infinite then \( a + b = b \).

**Proof:** Choose A, B disjoint so \( \#A = a, \#B = b \).

Let C \( \subseteq B \) such that \( \#C = \aleph_0 \) and let D = B - C. Let \( \#D = d \).

Then \( b = \aleph_0 + d \) so \( a + b = a + \aleph_0 + d = \aleph_0 + d = b \).

**Theorem 13:** If a is an infinite cardinal number then \( a + a = a \).

**Proof:** Choose A so that \( \#A = a \).

Let \( F = \{ \text{bijections } f : X \times 2 \rightarrow X \mid X \subseteq A \} \).

\( F \neq 0 \) (take X with \( \#X = \aleph_0 \)).

F is partially ordered by extension.

By Zorn’s Lemma \( \exists \) maximal \( f : X \times 2 \rightarrow X \) for some \( X \subseteq A \).

If \( A - X \) is infinite we contradict the maximality of \( f \) so \( A - X \) is finite.

\[ \#X + \#X = \#X \text{ and } \#A = \#X + \#(A - X) \]

\[ \#A + \#A = \#X + \#A - X + 2\#(A - X) \]

\[ = \#A + \#(A - X) \]

\[ = \#A. \]

**Theorem 14:** If \( a \leq b \) are cardinal numbers and \( b \) is infinite then \( a + b = b \).

**Proof:** Choose A, B with \( \#A = a, \#B = b \).

Since \( a \leq b, a + b \leq b + b = b \).

But \( b \leq a + b \) so \( a + b = b \).

**Theorem 15:** If a is an infinite cardinal number then \( a.a = a \).

**Proof:** Choose A so \( \#A = a \).

Let \( F = \{ \text{bijections } f : X \times X \rightarrow X \mid X \subseteq A \} \).

\( F \neq 0 \) (take X with \( \#X = \aleph_0 \)).

F is partially ordered by extension.

By Zorn’s Lemma \( \exists \) maximal \( f : X \times X \rightarrow X \) for some \( X \subseteq A \).

Let \( \#X = x \). Then \( x.x = x \).

Suppose \( x < a \).

Then \( \#(A - X) = a \) and so \( A - X \) has a subset, Y with \( \#Y = x \).

Then \( \#[(X \times Y) + (Y \times X) + (Y \times Y)] = 3x.x \)

\[ = x \] so

there exists a bijection from \((X \times Y) + (Y \times X) + (Y \times Y)\) to Y.

We can thus extend \( f \) to a bijection \( g : (X + Y) \times (X + Y) \rightarrow X + Y \), a contradiction.

Hence \( x = a \) and so \( a.a = a \).

At long last we can define a cardinal number as a set. A **cardinal number** is simply an ordinal number that is not equivalent to any of its predecessors.
Theorem 16: The class of ordinals, C, equivalent to a set \( S \) is itself a set.

Proof: Well order \( \wp(S) \) and let \( \gamma \) be the corresponding ordinal.
Then \( \alpha \in C \rightarrow \alpha < \gamma \rightarrow \alpha \in \gamma \).
So \( C = \{ \alpha \in \gamma \mid \alpha \approx S \} \).

Now that we have established that the set of ordinals equivalent to \( S \) is a set we can define, for any set \( S \) \#S to be the smallest ordinal equivalent to \( S \). Clearly it is a cardinal number.

We are now in a position to properly define the alephs, that is, to write every infinite cardinal as \( \aleph_\gamma \) for some ordinal \( \gamma \). Let \( \gamma \) be an infinite cardinal number.
Let \( S_\gamma \) be the set of infinite cardinal numbers that are less than \( \gamma \) (less than in the sense of cardinal numbers). \( S_\gamma \) is well-ordered by \( \leq \).
Let \( \beta \) be the ordinal number of this well-ordered set.
Then we denote \( \gamma \) by \( \aleph_\beta \).

Example 5: Let \( \gamma = \aleph_\omega \). Then \( S_\gamma = \{ \aleph_0, \aleph_1, \aleph_2, \ldots \} \) with
\( \aleph_0 < \aleph_1 < \aleph_2 < \ldots \)
The ordinal number of this well-ordered set is \( \omega \), which justifies the use of the notation \( \aleph_\omega \).

Example 6: Let \( \gamma = \aleph_{\omega+1} \). Then \( S_\gamma = \{ \aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega \} \) with
\( \aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega \)
The ordinal number of this well-ordered set is \( \omega + 1 \), which justifies the use of the notation \( \aleph_{\omega+1} \).

Theorem 17: If \( \alpha \) is an ordinal number and \( \#\alpha = \aleph_\beta \) then \( \aleph_\beta \leq \alpha < \aleph_{\beta+1} \).

Proof: \( \aleph_\beta \leq \alpha \) by definition of cardinals.
If \( \aleph_{\beta+1} \leq \alpha \) then \( \aleph_{\beta+1} \leq \aleph_\beta \), a contradiction.

Theorem 18: \( \gamma = \bigcup \{ \aleph_\alpha \mid \alpha < \beta \} \) is a cardinal number.

Proof: \( \gamma \) is an ordinal number.
Suppose \( \aleph_\delta = \#\gamma \) and \( \aleph_\delta < \gamma \).
Then \( \aleph_\delta \in \gamma \), so \( \aleph_\delta \in \aleph_\alpha \) for some \( \alpha < \beta \).
Then \( \aleph_\delta < \aleph_\alpha \leq \gamma \), a contradiction.

Theorem 19: \( \bigcup \{ \aleph_\alpha \mid \alpha < \beta^+ \} = \aleph_\beta \).

Proof: If \( \alpha < \beta^+ \) then \( \alpha \leq \beta \) and so \( \aleph_\alpha \subseteq \aleph_\beta \).

Theorem 20: If \( \beta \) has no predecessor then \( \bigcup \{ \aleph_\alpha \mid \alpha < \beta \} = \aleph_\beta \).

Proof: Let \( \aleph_\gamma = \bigcup \{ \aleph_\alpha \mid \alpha < \beta \} \).
For all \( \alpha < \beta \), \( \aleph_\alpha \subseteq \aleph_\beta \) and so \( \aleph_\gamma \subseteq \aleph_\beta \).
Hence \( \gamma \leq \beta \).
If \( \gamma < \beta \), \( \gamma + 1 < \beta \) so \( \aleph_{\gamma+1} \subseteq \aleph_\gamma \), a contradiction.