§7.1 Angles

We define the Euclidean Plane to be $\mathbb{R}^2$ where points are vectors. So we define a point to be a vector in $\mathbb{R}^2$ with the origin, $O$, defined as the zero vector. If $P_1$, $P_2$ are distinct points the line $P_1P_2$ is defined as the set $\{xP_1 + (1-x)P_2 | x \in \mathbb{R}\}$. Lengths can be defined in the usual Euclidean way. If $P = (x, y)$ the length of $P$ is defined to be $|P| = \sqrt{x^2 + y^2}$ and the distance between $P$ and $Q$ is defined to be $|P - Q|$. Most of the Euclidean Plane axioms are now easy to prove. One troublesome area is in the definition of angle. Euclid regarded angle intuitively. More recent accounts define angle in terms of arc length, but the definition of the length of a curve is some way off in our development.

We might decide to define the angle between the lines $OP$ and $OQ$ to be to be $\theta = \cos^{-1}\frac{P \cdot Q}{|P||Q|}$. This has the advantage that it can be easily extended to n-dimensional Euclidean Space. But there are many problems with such a definition. The main problem seems to be in showing that angles are additive.

If $A$, $B$ and $C$ are three points in the Euclidean plane and $\alpha$ is the angle between $OA$ and $OB$ and $\beta$ is the angle between $OB$ and $OC$ we might expect the angle, $\gamma$, between $OA$ and $OC$ to be $\alpha + \beta$. At least that ought to be true if the angles aren’t too big and rotating from $OA$ to $OB$ has the same orientation as from $OB$ to $OC$.

There are, of course problems of defining rotational orientation and dealing with the cases where going from $OA$ to $OB$ has the opposite orientation to going from $OB$ to $OC$. Then there are problems when the angles are both obtuse and the two rotations incorporate an entire rotation.

These problems could be dealt with, but there is the apparently difficult problem of proving, in the case where the rotations are in the same direction and do not involve less than one revolution, of showing that:

$$\cos^{-1}\frac{A \cdot C}{|A||C|} = \cos^{-1}\frac{A \cdot B}{|A||B|} + \cos^{-1}\frac{B \cdot C}{|B||C|}.$$
We will identify the Euclidean Plane with the Complex Plane and use the argument of a complex number to represent angles. If A, B, C are three points, regarded as complex numbers, we define the angle $\angle ABC$ to be $\arg(C - B) - \arg(A - B)$, an element of $\mathbb{R}_{2\pi}$, the system of real numbers modulo $2\pi$.

**Theorem 1:** The angles of a triangle total $2\pi$.

**Proof:** Let ABC be a triangle.

$\angle BAC = \arg(C - A) - \arg(B - A)$

$= \arg(A - C) + \pi - \arg(A - B) - \pi$

$= \arg(A - C) - \arg(A - B)$.

$\angle CBA = \arg(A - B) - \arg(C - B)$

$= \arg(A - B) - \arg(B - C) + \pi.$

$\angle ACB = \arg(B - C) - \arg(A - C)$.

Then $\angle BAC + \angle CBA + \angle ACB = \pi$.

**§7.2 The Rest of Mathematics**

The rest of Euclidean Geometry can now be built up. This can be extended to n-dimensions by considering the vector space of real vectors $(x_1, x_2, ..., x_n)$ with the usual inner product. We could define these vectors in a similar way to the way we defined ordered pairs, but that would be very clumsy. A vector $(x_1, x_2, ..., x_n)$ is simply a function from $\{1, 2, ..., n\}$ where $f(k)$ is written as $x_k$. Why didn’t we define ordered pairs in this way? The answer, of course, is that until we had defined ordered pairs we could not start to talk about relations or functions.

A $m \times n$ matrix is just a function from $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ with $f(i, j)$ written as $a_{ij}$. With abstract vector spaces our job is even easier because we don’t have to explain what the things are. They are just undefined entities subject to the vector space axioms. We could consider them as sets if we wanted to, but we don’t. The same thing is true of abstract algebra in general.

In fact the only area of mathematics where we might have problems in considering everything as a set is the area where we started – set theory itself. The universe of discourse is not a set, but a proper class – the class of all sets. True, the individual objects of study are sets but there are some concepts which might be hard to consider as a set. For example, the cardinal number of a set is its size, but once you go beyond finite sets you start to ask the question “what exactly is a cardinal number?” Is it a set? One answer is to choose it to be a particular set of a given size. But to justify doing this we would need the Axiom of Choice.

Our journey is over. We have learnt that everything theorem in mathematics can be proved, but you have to make some basic assumptions to begin with. Everything in mathematics can be considered as a set and its theory can be built on the ZF axioms for set
theory (perhaps supplemented by the Axiom of Choice and other optional extras). But those axioms have never been proved consistent and probably never will be.

Yet mathematics goes on in the confidence that if problems arise the ZF axioms will be modified. As in religion one “knows” the truth intuitively and does not need rely on creeds or sets of axioms, as useful as these are. And like religion, there is the unknowable in mathematics, statements that can never be proved true or false.

From here you can either go back to some other branch of mathematics that is less troublesome philosophically. Or you can read my notes on axiomatic set theory and go a little higher up the mountain.