5. REAL AND COMPLEX NUMBERS

§5.1 Fractions

The next stage in our development of the complex number system is to develop fractions or positive rational numbers. This mirrors the historical development of numbers because negative numbers came much later than fractions.

The concept of a fraction is quite sophisticated and it is no wonder than school students have so much trouble with them. For a start, what is a fraction? If you can do no better than talk about cutting up a cake into equal parts then your knowledge of fractions is somewhat undeveloped.

At first glance a fraction is a pair of natural numbers. But then we talk about equivalent fractions. Different pairs can represent the same number, as in \( \frac{6}{8} = \frac{3}{4} \). So fractions are more complicated than number pairs.

For a start we will exclude zero. We will bring it in at a later stage. So we begin by taking \( M = N - \{0\} \) to be the set of non-zero natural numbers. Then we form \( M \times M \), the set of ordered pairs \((m, n)\) of non-zero natural numbers.

Now we define the relation ~ on this set:

\[(a, b) \sim (c, d) \text{ if } ad = bc.\]

The first thing to do is to check that this is an equivalence relation, that is, it is reflexive, symmetric and transitive.

**Theorem 1:** The relation ~ is an equivalence relation.

**Proof:**

**Reflexive:** \((a, b) \sim (a, b)\) since \(ab = ba\).

**Symmetric:** Suppose \((a, b) \sim (c, d)\).

Then \(ad = bc\) and so \(cb = da\).

Hence \((c, d) \sim (a, b)\).

**Transitive:** Suppose \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\).

Then \(ad = bc\) and \(cf = de\).

Hence \((ad)(cf) = (bc)(de)\)

\[\therefore \ (af)(cd) = (be)(cd)\]

\[\therefore \ (a, b) \sim (e, f).\]

The set \( M \times M \) thus decomposes into equivalence classes and these are our fractions.

**Definition:** \( \frac{m}{n} \) is the equivalence class containing the ordered pair \((m, n)\).

Probably we should have written the equivalence classes as \([m, n]\) instead of the more familiar \( \frac{m}{n} \). The danger is to use things you learnt in primary school instead of justifying
them. So it is true that $\frac{6}{8} = \frac{3}{4}$, for example, but not because we “cancel top and bottom by 2” but because $6 \times 4 = 3 \times 8$.

Now we must define addition and multiplication, not of the ordered pairs, but of the equivalence classes. This involves what are called checks of well-definedness.

**Definition:** 
\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.
\]

This definition is not complete because the sum of the two equivalence classes is defined in terms of particular representatives. If \(\frac{a'}{b'} = \frac{a}{b}\) and \(\frac{c'}{d'} = \frac{c}{d}\) we naturally expect \(\frac{a'}{b'} + \frac{c'}{d'}\) to be equal to \(\frac{a}{b} + \frac{c}{d}\) but the definition is in terms of different numbers.

**Theorem 2:** Addition of fractions is well-defined.

**Proof:** Suppose \(\frac{a'}{b'} = \frac{a}{b}\) and \(\frac{c'}{d'} = \frac{c}{d}\).

Then \(a'b = b'a\) and \(c'd = d'c\).

Now \(\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}\) and 
\[
\frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd' + b'c'}{b'd'}.
\]

We need to check that \((ad + bc)(b'd') = (a'd' + b'c')(bd)\).

The LHS = \(adb'd' + bcb'd'\)
\[
= (b'a)(dd') + (d'c)(bb')
= (a'b)(dd') + (c'd)(bb')
= (a'd')(bd) + (b'c')(bd)
= RHS
\]

**Definition:** 
\[
\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.
\]

**Theorem 3:** Multiplication of fractions is well-defined.

**Proof:** Suppose \(\frac{a'}{b'} = \frac{a}{b}\) and \(\frac{c'}{d'} = \frac{c}{d}\).

Then \(a'b = b'a\) and \(c'd = d'c\).

Now \(\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}\) and 
\[
\frac{a'}{b'} \cdot \frac{c'}{d'} = \frac{a'c'}{b'd'}.
\]

We need to check that \((ac)(b'd') = (a'c')(bd)\).

The LHS = \(acb'd' = (b'a)(d'c) = (a'b)(c'd) = RHS\).
§5.2 The Arithmetic of Fractions

We now have to prove all the familiar properties of the positive rational numbers. These make constant use of the corresponding laws for natural numbers.

**Theorem 4**: Addition and multiplication of fractions is commutative and associative.

**Proof**: The commutative laws for addition and multiplication are obvious from the symmetry of the definitions. The associative laws are a little less obvious.

**Associativity of Addition**: \[ \frac{a}{b} + \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} + \frac{cf + de}{df} = \frac{adf + b(cf + de)}{bdf}. \]

**Associativity of Multiplication**: \[ \frac{a}{b} \cdot \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} \cdot \frac{cf + de}{df} = \frac{acf + ade}{b^2df}. \]

**Theorem 5**: Multiplication of fractions is distributive over addition.

**Proof**: \[ \frac{a}{b} \cdot \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} \cdot \left( \frac{cf + de}{df} \right) = \frac{acf + ade}{b^2df}. \]

In this case the ordered pairs are different. You probably would have justified the equality of these expressions by “cancelling by b” but why is this valid? Instead we observe that \( (acf + ade) \cdot b^2df = bdf(acbf + bdae) \)

and as a result \( (acf + de), bdf \sim (acbf + bdae, b^2df) \)

and hence \[ \frac{a(cf + de)}{bdf} = \frac{acf + ade}{b^2df}. \]

As we have constructed them the set of positive natural numbers is not a subset of the positive rational numbers. The former are finite sets while the latter are equivalence classes of pairs of natural numbers and hence are infinite sets. But we identify the positive natural number \( n \) with the rational number \( \frac{n}{1} \), which in turn consists of all the ordered pairs \((kn, k)\) for all positive natural numbers \( k \). But to justify this we need to check that the fractions of the form \( \frac{n}{1} \) behave like the natural numbers \( n \) themselves.

\[ \frac{m}{1} + \frac{n}{1} = \frac{m \cdot 1 + n \cdot 1}{1 \cdot 1} = \frac{m + n}{1} \quad \text{and} \]

\[ \frac{m}{1} \cdot \frac{n}{1} = \frac{m \cdot n}{1 \cdot 1} = \frac{mn}{1}. \]
§5.3 The Order Relation for Fractions

Definition: We define \( \frac{a}{b} \leq \frac{c}{d} \) if \( ad \leq bc \).

Theorem 6: The order relation is well-defined.

Proof: Suppose \( \frac{a'}{b'} = \frac{a}{b} \) and \( \frac{c'}{d'} = \frac{c}{d} \).

Then \( a'b = b'a \) and \( c'd = d'c \).

We must show that \( ad \leq bc \) if and only if \( a'd' \leq b'c' \).

Suppose \( ad \leq bc \).

Then \( (ad)(a'd') \leq (bc)(a'd') \).

But \( (bc)(a'd') = (a'b)(d'c) = (ab')(dc') = (ad)(b'c') \).

Hence \( (ad)(a'd') \leq (ad)(b'c') \) and so \( a'd' \leq b'c' \).

§5.4 The Positive Real Numbers

Having got the positive rational numbers we now turn our attention to positive real numbers. We cannot define these as numbers that represent points on the positive part of the real axis because this requires some geometric intuition. We could define them in terms of decimal expansions, though this would become clumsy when it comes to defining addition and multiplication. They are often defined as limits of convergent sequences of rational numbers. In keeping with our principle that every mathematical object has to be a set we shall define them as sets of rational numbers with certain properties.

Recall that a partial order on a set \( S \) is a relation that is reflexive, anti-symmetric and transitive. A partially ordered set is a set together with a partial order \( \leq \). An upper bound for a subset \( X \), if one exists, is an element \( u \in S \) such that \( \forall x \in X \rightarrow x \leq u \). A greatest element for \( X \), if it exists, is an upper bound for \( X \) that is an element of \( X \). Clearly, if it exists, it is unique. A subset \( X \) is an initial segment if \( \forall x \forall y[(x \in X) \land (y < x) \rightarrow (y \in S)] \).

Example 1: On the set \( Q^+ \) of positive rational numbers

\( X_1 = Q^+ \) has no upper bound;

\( X_2 = \{ x \mid x \text{ is a natural number} \} \) has no upper bound;

\( X_3 = \{ x \mid x^2 \leq 2 \} \) has upper bounds, such as 2, and 200, but it does not have a greatest element. Note that \( X_3 = \{ x \mid x^2 < 2 \} \) since our universe of quantification is the set of positive rational numbers and \( \sqrt{2} \) is irrational.

\( X_4 = \{ x \mid x^2 + 6 = 5x \} \) has a greatest element, namely 3. (The quadratic \( x^2 - 5x + 6 \) has two zeros, 2 and 3.)

\( X_5 = \emptyset \) has upper bounds (in fact every element) and no greatest element.

\( X_6 = \{ x \mid x^2 \leq 4 \} \) has upper bounds and a greatest element.

\( X_1, X_3, X_5 \) and \( X_6 \) are initial segments.

A positive real number is a non-empty set initial segment of positive rational numbers with an upper bound but no greatest element.
Example 2: Of the above subsets of $\mathbb{Q}^+$ only $X_3$ is a positive real number. This we will identify with the real number $\sqrt{2}$. $X_1$ has no upper bound, $X_5$ is empty and $X_6$ has a greatest element.

If $X$, $Y$ are positive real numbers we define $X + Y$, $XY$ and $X \leq Y$ in terms of their elements as follows:

$$X + Y = \{x + y \mid x \in X, y \in Y\};$$
$$XY = \{xy \mid x \in X, y \in Y\};$$
$$X \leq Y \text{ iff } X \subseteq Y.$$

Theorem 7: If $X$, $Y$ are positive real numbers so are $X + Y$ and $XY$.

Proof: Clearly $X + Y$ and $XY$ are non-empty.

If $u$ is an upper bound for $X$ and $v$ is an upper bound for $Y$ then $u + v$ and $uv$ are clearly upper bounds for $X + Y$ and $XY$ respectively.

Let $x + y \in X + Y$. Since $X$, $Y$ have no maximum elements, there exist $x_1 \in X$ and $y_1 \in Y$ such that $x < x_1$ and $y < y_1$. Hence $x + y < x_1 + y_1 \in X + Y$. Similarly for multiplication.

Finally we must show that $X + Y$ and $XY$ are initial segments. Let $u = x + y$ where $x \in X$ and $y \in Y$ and let $v < u$.

Multiplication is a little trickier. Let $u = xy$ and let $w = u - v > 0$.

We want to find $x_1$ and $y_1$ such that $v = x_1 y_1$ and $0 < x_1 < x$ and $0 < y_1 < y$.

Write $x_1 = x - a$ and $y_1 = y - b$.

We want $v = (x - a)(y - b) = xy - (ay + bx) + ab$

$= u - (ay + bx) + ab$

so $w = ay + bx - ab$.

If we chose any $b$ and defined $a = \frac{w - bx}{y - b}$ then $ay + bx - ab$ would equal $w$ and so, working back, $v = (x - a)(y - b) \in XY$.

This would be fine over $\mathbb{Q}$, provided $b \neq y$ but since we are working over $\mathbb{Q}^+$ we must ensure that $w - bx$, $x - a$ and $y - b$ are all positive. The fact that $v$ and $y - b$ are positive will ensure that $x - a$ is positive so we only need to worry about $w - bx$ and $y - b$.

We can achieve this by taking $b = \text{MIN}\left(\frac{w}{2x}, \frac{y}{2}\right)$ and $a = \frac{w - bx}{y - b}$. We can verify that indeed $v = (x - a)(y - b) \in XY$.

The usual associative, commutative and distributive properties can easily we deduced from the corresponding properties for positive rational numbers. We can also show that for positive rational numbers $q$ the corresponding positive real numbers $\{x \mid x < q\}$ behave exactly like the positive real numbers themselves and so we associate $q$ with $\{x \mid x < q\}$. So, within the positive real numbers we have a copy of the positive rational numbers.

Hence the multiplicative identity of the positive reals is $\{x \mid x < 1\}$ which we associate with the rational number 1. And the positive real $\{x \mid x^2 < 2\}$ has the property that its square is $\{x \mid x < 2\}$ which we associate with the rational number 2. So we denote $\{x \mid x^2 < 2\}$ by $\sqrt{2}$ and note that indeed it is the (positive) square root of 2.
There are many other properties that ought to be investigated. But many of these follow from the completeness of the positive reals, the fact that every non-empty subset of positive real numbers that has an upper bound has a least upper bound. This is easy! If \( S \) is a non-empty set of real numbers bounded above then \( \cup S \) is its least upper bound. But now we move on – to negative numbers.

§5.5 Zero and Negative Real Numbers

The construction that takes us from the positive real numbers to the whole real line is similar to that that took us from the natural numbers to the positive rationals. We take pairs of real numbers and define an equivalence relation. The real numbers are simply the equivalence classes.

The pair \((a, b)\) will ultimately represent \(a - b\) which, depending on the relative sizes of \(a, b\) can give positive, negative or zero real numbers. But because the representation as \(a - b\) is not unique we must take equivalence classes.

So take the set \(\mathbb{R}^+ \times \mathbb{R}^+\) to be the set of all pairs of positive real numbers and define the relation \(\sim\) as follows:

\[(a, b) \sim (c, d) \text{ iff } a + d = b + c.\]

I leave it as an exercise to prove that this is indeed an equivalence relation. Let \([x, y]\) denote the equivalence class containing \((x, y)\). So if \(a + d = b + c\) then \([a, b] = [c, d]\).

Define addition, multiplication and ordering by:

\begin{align*}
[a, b] + [c, d] &= [a + c, b + d]; \\
[a, b] \cdot [c, d] &= [ac + bd, ad + bc]; \\
[a, b] &\leq [c, d] \text{ if } a + d \leq b + c.
\end{align*}

These definitions are motivated by the fact that:

\begin{align*}
(a - b) - (c - d) &= (a + c) - (b - d), \\
(a - b)(c - d) &= (ac + bd) - (ad + bc) \text{ and} \\
a - b &\leq c - d \text{ if and only if } a + d \leq b + c.
\end{align*}

You know what lies ahead. We must show that these operations are well-defined and that the associative, commutative properties hold as well as all the other elementary properties. We don’t need to actually do this, except as an exercise. The main thing is to be convinced that it can be done!

We might be tempted to identify the positive real numbers with those real numbers of the form \([x, 0]\) but, of course 0 is not positive. Instead we must identify \([x + 1, 1]\) with \(x\).

For example:

\begin{align*}
[x + 1, 1] + [y + 1, 1] &= [x + y + 2, 2] \\
&= [x + y + 1, 1] \text{ since } (x + y + 2) + 1 = (x + y + 1) + 2 \text{ and} \\
[x + 1, 1], [y + 1, 1] &= [(x + 1)(y + 1) + 1, (x + 1) + (y + 1)] \\
&= [xy + x + y + 2, x + y + 2] \\
&= [xy + 1, 1].
\end{align*}
§5.6 Complex Numbers

We are almost there. The last stage in the development of our number system is to extend the real numbers to the system of complex numbers, \( \mathbb{C} \). This is by far the easiest stage of all. We define a complex number as a pair of real numbers. There is no equivalence relation to be defined. There are no equivalence classes. Just the pairs themselves, which we will write as \([x, y]\) instead of the usual \((x, y)\) to make it look more like the previous stages.

A complex number is a pair of real numbers \([x, y]\). Keeping in mind that we will eventually identify this with \(x + iy\), where \(i^2 = -1\), we make the following definitions:

\[
[a, b] + [c, d] = [a + c, b + d];
[a, b].[c, d] = [ac - bd, ad + bc];
\]

Note that we do not define an ordering. There is no order relation that is consistent with these algebraic operations.

We identify the complex numbers of the form \([x, 0]\) with the corresponding real number \(x\) and not that they behave the same. \([x, 0] + [y, 0] = [x + y, 0 + 0] = [x + y, 0]\) and \([x, 0].[y, 0] = [xy - 0, 0 + 0] = [xy, 0]\).

The multiplicative identity is \([1, 0]\), which we identify with the real number 1 and we define \(i = [0, 1]\). Note that \(i^2 = [-1, 0]\) which we write as \(-1\).

The fact that \([x, y] = [x, 0] + [0, 1].[y, 0]\) means we can identify \([x, y]\) with \(x + iy\).

All that remains is to check out the basic properties of complex numbers and we are finished. But probably by now you are thoroughly bored with the whole process and are quite happy just accepting the assurance “believe me, it works!”