§5. PFISTER FORMS

§5.1. Pfister Forms

The tensor product of two diagonal quadratic forms is defined by:
\[ \langle a_1, \ldots, a_m \rangle \otimes \langle b_1, \ldots, b_m \rangle = \langle a_1b_1, \ldots, a_1b_m, a_2b_1, \ldots, a_2b_m, \ldots, a_mb_m \rangle. \]

NOTES:
1. \(\otimes\) is well-defined, commutative, associative and distributive over \(\oplus\).
2. In \(W(F)\) \(ab = \text{core} (a \otimes b)\).
3. \(a \otimes H\) is hyperbolic for any quadratic form \(a\).

A Pfister plane is a 2-dimensional quadratic form of the form \(\langle 1, a \rangle\).

Example 1: \(H = \langle \langle -1 \rangle \rangle\) is a Pfister plane.

A Pfister form is a quadratic form of the form \(\langle \langle a_1 \rangle \rangle \otimes \cdots \otimes \langle \langle a_n \rangle \rangle\).

Examples 2:
1. \(2^nH = \langle \langle -1, -1, \ldots, -1 \rangle \rangle\) (with \(n + 1\) terms).
2. The quadratic form \(\langle 1, -a, -b, ab \rangle\) associated with the quaternion algebra \([a, b]_F\) is \(\langle \langle -a, -b \rangle \rangle\).
3. \(2^n\langle 1 \rangle = \langle \langle 1, 1, \ldots, 1 \rangle \rangle\) (with \(n\) terms).

NOTE: \(\langle -1, a_1, \ldots, a_n \rangle = H \otimes \langle \langle a_1, \ldots, a_n \rangle \rangle\) and so is hyperbolic.

If \(\varphi\) is a quadratic form, \(\varphi = \langle 1 \rangle \oplus \varphi'\) for some quadratic form \(\varphi'\). Call \(\varphi'\) the pure subform of \(\varphi\). This generalises the pure part of a quaternion algebra.

If \(\varphi\) is a quadratic form:
\(D_F(\varphi)\) is defined to be \(\{ a \in F \mid \varphi(x_1, \ldots, x_n) = a \text{ for some } n \text{ and some } x_1, \ldots, x_n \in F \}\).
\(D^\#_F(\varphi)\) denotes the non-zero elements of \(D_F(\varphi)\).

Example 3: \(\frac{1}{2} \in D_Q(\langle 1, 1 \rangle)\) since \(\frac{1}{2} = (\frac{1}{2})^2 + (\frac{1}{2})^2\).

Theorem 1:
1. \(\langle \langle a, b \rangle \rangle = \langle \langle a, ab \rangle \rangle\).
2. If \(a + bk^2 \neq 0\) then \(\langle \langle a, b \rangle \rangle = \langle \langle a + bk^2, ab \rangle \rangle\).

Proof:
1. \(\langle \langle a, ab \rangle \rangle = \langle 1, a, ab, a^2b \rangle = \langle 1, a, ab, b \rangle = \langle \langle a, b \rangle \rangle\).
2. Suppose that \(a + bk^2 \neq 0\).
Then \(\langle a, b \rangle \cong a(x - kby)^2 + b(kx + ay)^2 = \langle a + bk^2, (a + bk^2)ab \rangle\).
(Note that the determinant of the transformation is \(a + bk^2 \neq 0\).)
Hence \(\langle \langle a, b \rangle \rangle = \langle 1, a, b, ab \rangle = \langle 1, a + bk^2, (a + bk^2)ab, ab \rangle = \langle \langle a + bk^2, ab \rangle \rangle\).

Corollary: If \(a + b \neq 0\) then \(\langle \langle a, b \rangle \rangle = \langle \langle a + b, ab \rangle \rangle\).
§5.2. Isotropic Pfister Forms

Theorem 2: If $\varphi = \langle a_1, \ldots, a_n \rangle$ and $b_1 \in D_F(\varphi)^\#$ then $\varphi \equiv \langle b_1, \ldots, b_n \rangle$ for some $b_2, \ldots, b_n \in F$.

Proof: Induction on $n$. 
If $n = 1$, $\varphi = \langle a_1 \rangle$, $\varphi' = \langle a_1 \rangle$. Hence $b_1 = a_1 x^2$ for some $x \in F^*$ and so $\varphi = \langle 1, a_1 \rangle = \langle 1, b_1 \rangle = \langle \langle b_1 \rangle \rangle$.

Suppose it is true for $n - 1$.
Let $\tau = \langle a_2, \ldots, b_{n-1} \rangle$.
Now $\varphi = \langle a_1 \rangle \otimes \tau$

$= \langle 1, a_1 \rangle \otimes \tau$

$= \langle (1) \otimes (a_1) \rangle \otimes \tau$

$= \tau \otimes a_1 \tau$

$= \langle 1 \rangle \otimes \tau' \otimes a_1 \langle 1 \rangle \otimes \tau'$.

Hence $\varphi' = \tau' \otimes a_1 \langle 1 \rangle \otimes \tau'$.

Hence $b_1 = x + a_1 (k^2 + y)$ for some $x, y \in D_F(\tau')$.

Case I: $k^2 + y = 0$:
Then $b_1 \in D_F(\tau')$ and so by induction,
$\tau = \langle b_1, \ldots, b_{n-1} \rangle$ for some $b_2, \ldots, b_{n-1}$.
Putting $b_n = a_1$, $\varphi = \langle b_1, \ldots, b_n \rangle$.

Case II: $k^2 + y \neq 0$:
In this case we show first that $\varphi = \langle a_1(k^2 + y) \rangle \otimes \tau$.

If $y = 0$, this is obvious, so suppose $y \neq 0$.
Then $y \in D_F(\tau')$.
By induction, $\tau = \langle x, b_3, \ldots, b_n \rangle$ for some $b_3, \ldots, b_n$.
Thus $\varphi = \langle a_1(k^2 + y), x, b_3, \ldots, b_n \rangle$.
Now $\langle a_1(k^2 + y), x \rangle = \langle a_1(k^2 + y) + x, a_1(k^2 + y)x \rangle$ by the corollary to Theorem 1.

$= \langle b_1, b_2 \rangle$ if we choose $b_2 = a_1 (k^2 + y)$.

Hence $\varphi = \langle b_1, b_2, b_3, \ldots, b_n \rangle$.

Theorem 3: An isotropic Pfister form is hyperbolic.

Proof: If $\varphi$ is an isotropic Pfister form, $\varphi = H \oplus \theta$ for some $\theta$.
Therefore $\varphi' = \langle -1 \rangle \oplus \theta$.
Hence $-1 \in D_F(\varphi)^\#$ and so by Theorem 2,
$\varphi = \langle -1, b_2, \ldots, b_n \rangle$ for some $b_2, \ldots, b_n$ which is hyperbolic.

§5.3. The Characteristic of a Witt Ring

The characteristic of a ring (with 1) is [the additive order of 1 if this is finite]

0 if 1 has infinite order

NOTE: If $R$ has characteristic $n$ then the additive order of every element divides $n$, for $nr = (n1)r = 0$.

The characteristic of a field is 0 or prime, but in general it may be composite.
Theorem 4: If $W(F)$ has finite characteristic it is a power of 2.
Proof: Let $\text{char } W(F) = n$ where $2^{k+1} < n \leq 2^k$.
Then $n \langle 1 \rangle$ is hyperbolic and so $2^k$ is isotropic.
Hence by Theorem 3, it is hyperbolic and so in $W(F)$, $2^k \langle 1 \rangle = 0$.
Hence $n$ divides $2^k$ and so is a power of 2.

Theorem 5: If $-1$ can be expressed as a sum of $n$ squares in $F$, but no fewer, then $n$ is a power of 2.
Proof: Let $2^k \leq n < 2^{k+1}$.
Then $-1 = x_1^2 + \ldots + x_n^2$ for some $x_i$’s $\in F^\#$.
Therefore $(n + 1) \langle 1 \rangle$ is isotropic and so $2^{k+1} \langle 1 \rangle$ is hyperbolic.
Thus in $W(F)$ $2^{k+1} \langle 1 \rangle = 0$.
Hence $\text{char}(F)$ divides $2^{k+1}$.
Let $\text{char}(F) = 2^{t+1}$ where $0 \leq t \leq k$.
Then $2^{t+1} \langle 1 \rangle$ is hyperbolic and so $2^{t+1} \langle 1 \rangle \cong 2^t \langle 1, -1 \rangle$.
By Witt’s Cancellation Theorem $2^t \langle 1 \rangle \cong 2^t \langle -1 \rangle$.
Hence $-1$ can be expressed as a sum of $2^t$ squares and so $2^k \leq n \leq 2^t \leq 2^k$.
Thus $n = 2^t = 2^k$.

§5.4. The Level of a Field

The level of a field is
\[
\begin{cases} 
\text{the smallest } n \text{ such that } -1 \text{ is a sum of } n \text{ squares} & \text{if } \text{no such } n \text{ exists} \\
\infty & \text{otherwise} 
\end{cases}
\]

Theorem 6: If $F$ has finite level $n$, $\text{char } W(F) = 2n$.

Examples 3:
(1) level $\mathbb{C} = 1$;
(2) level $\mathbb{R} = \text{level } \mathbb{Q} = \infty$;
(3) the level of a finite field $= \begin{cases} 1 & \text{if } |F| \equiv 1 \pmod{4} \\
2 & \text{if } |F| \equiv 3 \pmod{4} 
\end{cases}$.

Theorem 7: If $n$ is a positive square-free integer the level of $\mathbb{Q}[\sqrt{n}] \leq 4$.
Proof: $-1 = (i \sqrt{n})^2 + n - 1$.
By a theorem of Lagrange, every positive integer is the sum of at most 4 squares.
Hence level $F \leq 5$.
By Theorem 5, level $F \leq 4$.

In fact, level $\mathbb{Q}[\sqrt{n}] = \begin{cases} 1 & \text{if } n = 1 \\
4 & \text{if } n \equiv -1 \pmod{8} \\
2 & \text{otherwise} 
\end{cases}$.

Example 4: Express $-1$ as a sum of 4 squares in $\mathbb{Q}[\sqrt{7}]$.
Solution: $-1 = (i \sqrt{7})^2 + 2^2 + 1^2 + 1^2$. 

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Example 5: Express −1 as a sum of 2 squares in \( \mathbb{Q}[i\sqrt{3}] = \mathbb{Q}[\omega] \).
Solution: \(-1 = \omega + \omega^2 \).

Example 6: Express −1 as a sum of 2 squares in \( \mathbb{Q}[i\sqrt{13}] \).
Solution: \(-1 = \left(\frac{3}{2}\right)^2 + \left(\frac{i\sqrt{13}}{2}\right)^2 \).

NOTE: If \( n = a^2 + b^2 \), \(-1 = \left(\frac{i\sqrt{n}}{a}\right)^2 + \left(\frac{b}{a}\right)^2 \).
If \( n - 1 = a^2 + b^2 \) then \(-1 = \left(\frac{a}{n-1} + \frac{b}{n-1}\right)^2 + \left(\frac{b}{n-1} - \frac{a}{n-1}\right)^2 \).