14. NILPOTENT GROUPS

§14.1. The Ascending Central Series
The centre of a group $G$ is $Z(G) = \{ z \in G \mid zg = gz \text{ for all } g \in G \}$. We can use this to define a whole series of “centres”, called the ascending central series.

We define $Z_i(G)$ inductively by defining $Z_0(G) = 1$ and $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$.

So $Z_1(G) = Z(F)$ and $Z_i(G) = \{ z \in G \mid [z, g] \in Z_{i-1}(G) \}$ for all $i \geq 1$.

The ascending central series is thus:

$1 \leq Z(G) \leq Z_2(G) \leq Z_3(G) \leq \ldots$

We define $G$ to be nilpotent if $Z_n(G) = G$ for some $n$, and the smallest such $n$ is called the nilpotency class, or just the class. The trivial group is the only nilpotent group of class 0 and non-trivial abelian groups are nilpotent groups of class 1.

Example 1: The dihedral group of order 8 is nilpotent of class 2.
If $G = \langle A, B \mid A^4, B^2, [A, B] = A^2 \rangle$ we have $Z(G) = \langle A^2 \rangle$ and $Z_2(G) = G$.

Example 2: Generalizing example 1, if $G = D_{2n+1} = \langle A^{2n}, B^2, [A, B] = A^2 \rangle$ is the dihedral group of order $2^{n+1}$ then for $r < n$, $Z_r(G) = \langle A^{2^{n-r}} \rangle$ and $Z_n(G) = G$. Hence $G$ is nilpotent of class $n$.

Theorem 1: Finite $p$-groups are nilpotent.
Proof: $Z_{r+1}(G)/Z_r(G) = Z(G/Z_r(G))$. Since the centre of a non-trivial $p$-group is non-trivial, $Z_r(G) < Z_{r+1}(G)$ unless $Z_r(G) = G$.

Theorem 2: Subgroups and quotient groups of nilpotent groups are nilpotent.
Proof: Suppose $Z_n(G) = G$ and let $H \leq G$. For each $r$, $Z_r(H) \geq Z_r(G) \cap H$ and so $Z_n(H) = H$.
Suppose now that $H$ is normal in $G$. For each $r$, $Z_r(G)/H \geq Z_r(G/H)$ and so $Z_n(G/H)$ is trivial.

Clearly, if $G$ is nilpotent of class $n$ then the nilpotency class of every subgroup and every quotient group of $G$ is less than or equal to $n$.

Theorem 3: Let $G$ be a nilpotent group of class $n$ and let $H \leq G$.
Let $H_0 \leq H_1 \leq H_2$ be defined by: $H_0 = H$, $H_{i+1} = N_G(H_i)$ for all $i \geq 0$.
Then $H_n = G$.
Proof: $Z_0(G) \leq H_0$. We prove by induction on $r$ that $Z_r(G) \leq H_r$.
For $z \in Z_{r+1}(G)$ and $h \in H_r$, $[z, h] \in Z_r(G)$ and so $z^{-1}h^{-1}z \in H_rZ_r(G) \leq H_r$.
Hence $z \in N_G(H_r) = H_{r+1}$. Thus $Z_{r+1}(G) \leq H_{r+1}$.

Corollary 1: If $G$ is nilpotent and $H \leq G$ then $H < N_G(H)$.
Proof: If $H = N_G(H)$ then $H_r$, as defined above, will be equal to $H$ for all $r$.

Corollary 2: Every maximal subgroup of a nilpotent group is normal.

Corollary 3: Every maximal subgroup of a nilpotent group has finite, prime index.
Proof: If $M$ is a maximal subgroup then $G/M$ has no proper subgroups and so is isomorphic to $C_p$ for some prime $p$. 


Theorem 4: A finite group is nilpotent if and only if it is a direct product of its Sylow subgroups.

Proof: Finite $p$-groups are nilpotent, and hence so is any direct product of $p$-groups.

Now suppose that $G$ is a finite nilpotent group and let $P$ be any Sylow subgroup of $G$.

Let $N = N_G(P)$. Suppose $N < G$. Then $N < N_G(N)$, by Theorem 4 (Cor 1).

Let $x \in N_G(N) - N$.

Hence $x^{-1}Px \leq x^{-1}N = N$ and so $x^{-1}Px$ is a Sylow subgroup of $N$, as is $P$.

It follows that $y^{-1}x^{-1}Pxy = P$ for some $y \in N$ and so $xy \in N$. But $y \in N$ and $x \in N$, a contradiction.

It must therefore be that $N = G$ and so $P$ is normal in $G$.

$G$ is therefore the direct product of its Sylow subgroups.

§14.2. Nilpotent Groups of Class 2

Nilpotent groups of class 2 have many properties in common with abelian groups. In an abelian group we have $(xy)^n = x^n y^n$ for all $x, y$. If the group has class 2 there is a similar, but slightly more complicated result.

Theorem 5: If $G$ is nilpotent of class 2 then $(xy)^n = x^n y^n [y, x]^{1/2n(n-1)}$.

Proof: We prove this by induction on $n$. For $n = 1$ it is obvious.

Suppose it is true for $n$.

Then $(xy)^{n+1} = (xy)^n(xy) = x^n y^n [y, x]^{1/2n(n-1)}.xy$.

Since $[y, x] \in Z(G)$ we can write this as $x^n y^n [y, x]^{1/2n(n-1)}$.

Now $yx = xy[y, x]$ so that each time we bring an $x$ to the left, past a $y$, we introduce a factor of $[y, x]$. These factors can be moved together with all the others, at the end of the expression.

Hence $y^n x = xy^n [y, x]^n$ and so $(xy)^{n+1} = x^{n+1} y^{n+1} [y, x]^{1/2n(n-1) + n}$

$= x^{n+1} y^{n+1} [y, x]^{1/2n(n+1)}$. So it is true for all $n$.

As with infinite abelian groups we define the torsion subgroup of $G$ as

$t(G) = \{ g \in G \mid g^n = 1 \text{ for some } n > 0 \}$

We call $G$ a periodic group if $G = t(G)$ and torsion-free if $t(G/t(G))$ is trivial. But first we have to show that $t(G)$ is indeed a normal subgroup. Actually the normality is obvious since conjugates have the same order. The one thing that is not true in general, but is true for abelian groups and nilpotent groups of class 2, is closure.

Theorem 6: If $G$ is nilpotent of class 2, $t(G)$ is a normal subgroup of $G$.

Proof: Suppose that $x, y \in t(G)$ and that $x^n = y^n = 1$. Without loss of generality we can take the same power for each, so assume that $x^n = y^n = 1$.

Now $x^{-1}yx = y[y, x]$. Since both factors commute with $y$ (the latter because $[y, x] \in Z(G)$) we have that $x^{-1}yx$ commutes with $y$. Hence, for all $k$, $[y, x]^n = [y^{-1}, x^{-1}yx]^n = y^{-n}(x^{-1}yx)^n = 1$.

Hence $(xy)^{2n} = 1$. (We need the extra 2 in case $n$ is even.)

We define the Sylow $p$-subgroup, $Syl_p(G)$, of a nilpotent group of class 2 as we do for abelian groups. $Syl_p(G) = \{ g \in G \mid g^n = 1 \text{ for some } n \}$. 

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**Theorem 7:** If $G$ is a nilpotent group of class 2 then, for each prime $p$, $\text{Syl}_p(G)$ is a normal subgroup of $G$.

**Proof:** The fact that $\text{Syl}_p(G)$ is normal is obvious, since conjugates have the same power. In fact the only fact that requires checking is closure.
Suppose $g, h \in \text{Syl}_p(G)$. Suppose $g^n h^m = 1$ for some $m, n$. Without loss of generality we may assume that $g^n h^m = 1$.
Then $(xy)^p = x^p y^p [y, x]^{1/2 p^{n+1} (p^{n+1} - 1)} = [y, x]^{1/2 p^{n+1} (p^{n+1} - 1)} = 1$.
(For odd $p$ the power $p^n$ would have been sufficient, but for $p = 2$ we need the extra factor to compensate for the $1/2$.)

**Theorem 8:** A periodic nilpotent group of class 2 is a direct product of its Sylow subgroups.

**Proof:** We prove this in a similar manner to the case of periodic abelian groups.

A group $G$ is **verbally abelian** if there exists a word $W(x, y)$ in two variables such that $(G, \star)$ is an abelian group under the operation $x \star y = W(x, y)$.

When a group is verbally abelian we have two group structures on the same set. Suppose $G$ is the original group and $G_\star$ is the abelian group on the set $G$. Then subgroups of $G$ are subgroups of $G_\star$. The order of elements is the same in both groups. Any automorphism of $G$ is an automorphism of $G_\star$.

**Theorem 9:** Suppose $G$ is a nilpotent group of class 2 and $G'$ has finite odd exponent then $G$ is verbally abelian.

**Proof:** Let $G$ be nilpotent of class 2 and suppose that $G$ has exponent $n$, where $n$ is odd. Then it is easily checked that for all $k$, $(G, \star)$ is a group where $x \star y = xy[x, y]^k$.

Moreover, since $y \star x = y[x, y]^k = xy[y, x]^{k+1} = xy[x, y]^{-k-1}$, $(G, \star)$ is abelian if $k = \frac{n - 1}{2}$.