9. RULER AND COMPASS CONSTRUCTIONS

§9.1. Ruler and Compass Constructions

Many geometric constructions can be carried out with just two tools – a ruler and a compass (and, of course, a sharp pencil!) The classic examples are bisection – bisection of intervals and of angles. Strictly speaking, instead of a “ruler” we should be talking about a “straight edge”, because a ruler can be used to make measurements, whereas the only use that we allow for the straight edge is the drawing of a straight line between distinct points.

Although many ruler and compass constructions have a practical use, we’re taking here a purely theoretical approach where we use idealised instruments. Our pencil is assumed to be infinitely sharp, our straight edge is a perfect straight line and our compass draws arcs of perfect circles.

A ruler and compass construction is carried out in the Euclidean Plane. We begin with a finite set of points which we call “marked” points because they have been specifically identified. As we proceed with our constructions we mark additional points. Although our pencil produces infinitely many additional points when we draw a line, only those that occur as intersections in the construction will be considered as marked.

At each stage in the construction we’ll have a finite set of marked points. We can extend the construction as follows:

- Draw the line through any two distinct marked points. (In practice this line would be a line segment, with two endpoints but in theory the lines would extend indefinitely.)

- Draw the circle with one marked point as centre, passing through another marked point.

Having additional marked points in our construction we can now draw more lines and more circles and so continue the construction as far as we wish. Note that we don’t have to include all possible lines or circles at each stage in our construction. We usually have a goal and we choose just what we need to construct to achieve our goal.

**Example 1:** Bisect an interval by ruler and compass.

**Solution:** Suppose the endpoints of the interval are P and Q.

1. With centre P draw the circle passing through Q.
2. With centre Q draw the circle passing through P.

Suppose that the two points of intersection of these circles are F and G.

3. Draw FG. The point of intersection, M, of PQ and AB is the required midpoint.
This is, in fact, the perpendicular bisector. The line $AB$ will be perpendicular to the original line $PQ$. The proof that $PM = MQ$ and that $AB \perp PQ$ uses the elementary geometry of congruent triangles.

Note that $\Delta APQ$ is equilateral, so this is also a construction for the equilateral triangle on a given base.

**Example 2:** Bisect an angle by ruler and compass.

**Solution:**

![Diagram of angle bisector construction]

The Greeks considered arithmetic and geometry as being two different ways of looking at the same number system, and geometrical constructions to perform arithmetic operations were considered very natural. Since the only numbers they could conceive of arithmetically were rational numbers they assumed that that’s all that could be obtained geometrically.

So it came as a great shock for them to discover that $\sqrt{2}$ is irrational, that is a number that had a proper geometric existence did not have a proper arithmetic one. But then irrational numbers came to be accepted though they didn’t lose their geometric flavour. Number was a geometric concept. And it was accepted as an unwritten axiom that only numbers which arose in the context of geometric constructions could possibly exist.

Certain problems were posed where a construction “must exist but is hard to find”. The three most famous were the “doubling of a cube”, the “trisection of an angle” and the “squaring of a circle”. The numbers involved in these three problems clearly exist, they thought (it’s interesting that their intuition must have been using some primitive notion of continuity – a concept that took another couple of thousand years to become fully developed), so clearly there had to be a corresponding construction. And “construction” meant a construction using ruler and compass.

**Doubling the Cube:**

There’s a legend that, when asked for a way of stopping a plague that was attacking the city of Delos, the Oracle of Delphi advised that the altar of Apollo should be doubled in size. The altar was in the shape of a cube and although its sides were doubled the plague continued. The Oracle then revealed that the citizens of Delos had not done as instructed since they had increased the altar eight-fold. What was required was to double the volume.

The problem of doubling the cube is thus:

**Given the side of a cube, construct the side of a cube with twice the volume.**

**Trisecting an Angle:**

**Divide any given angle into three equal pieces.**

For certain angles, such as $90^\circ$, it can easily be done (a $30^\circ$ angle can easily be constructed). But the problem is to do it for any given angle.
Squaring the Circle:

Construct a square whose area is exactly equal to that of a given circle.

The methods allowed in all these constructions are the use of a ruler and compass. The compass is to be used to draw circles through given points and passing through others. The ruler must be used solely as a straight-edge for joining points by straight lines, not for measurement. For this reason, ruler and compass constructions are often called constructions by *straight-edge* and compass.

The reason for disallowing measurement is the question of accuracy. The accuracy with which we can measure lengths is limited by the scale of the markings. Even if we had the means to magnify the scale we’d have to end up making judgements. The whole philosophy of ruler and compass constructions is to have a procedure that is theoretically exact.

There do exist ruler and compass methods for getting quite good approximations to the solutions to all three problems. But that’s not the point. With a question of existence it’s no good saying that these numbers *approximately* exist. We need methods that are *exact*. And it’s been shown that all three problems are insoluble by ruler and compass methods. The geometric concept of number in terms of constructability thus proved to be inadequate and so the more general concept of real number gradually emerged.

The objects in a ruler and compass construction are points (denoted A, B, C, ...) and lines (denoted by a, b, c ...). Lines include straight lines and circles, which can either be given, or can be drawn in the course of the construction. Other curves, such as parabolas, can only occur if they’re given at the beginning of the construction.

LINE(A, B) = the line through A,B (where A ≠ B)

CIRCLE(A, B) = the circle with centre A, passing through B (where A ≠ B)

INTERSECTIONS of lines with lines.

A = p ∩ q indicates that A is the point of intersection of straight lines p and q or one of the points of intersection when one of p, q is a circle.

A, B = p ∩ q indicates that A and B are the distinct points of intersection of lines p and q where one of them is a circle.

A ∈ p describes A as an arbitrary points on p (distinct from any other marked point).

A, B ∈ p describes A, B as distinct arbitrary points on p.

§9.2. Examples of Ruler and Compass Constructions

The following are some examples of some standard ruler and compass constructions.

(1) EQUILATERAL(A, B) is a point C that makes ABC an equilateral triangle

c = CIRCLE(A, B)
d = CIRCLE(B, A)
C = d ∩ e
(2) **BISECT(A, B)** is the perpendicular bisector, b, of AB.

- \(c = \text{CIRCLE}(A, B)\)
- \(d = \text{CIRCLE}(B, A)\)
- \(C, C' = c \cap d\)
- \(b = \text{LINE}(C, C')\)

(3) **MIDPOINT(A, B)** is the midpoint, M, of the interval AB.

- \(a = \text{LINE}(A, B)\)
- \(b = \text{BISECT}(A, B)\)
- \(M = a \cap b\)

(4) **REFLECT(A, B)** is the reflection, A’, of A in B.

- \(a = \text{LINE}(A, B)\)
- \(c = \text{CIRCLE}(B, A)\)
- \(A' = a \cap c\)

(5) **PERPENDICULAR(A, B)** is the line through B that's perpendicular to AB

- \(A' = \text{REFLECT}(A, B)\)
- \(\text{PERPENDICULAR}(A, B) = p = \text{BIS}(A, A')\)

(6) **PARALLELOGRAM(A, B, C)** is the point D such that ABCD is a parallelogram

- \(M = \text{MIDPOINT}(A, C)\)
- \(D = \text{REFLECT}(B, M)\)

(7) **PARRALLEL(A, m)** is the line, p, through A that's parallel to m

- \(B, C \in m\)
- \(D = \text{PARALLELOGRAM}(C, B, A)\)
- \(p = \text{LINE}(A, D)\)
(8) **BISECT**(a, b) is the bisector, m, of one of the angles formed between straight lines a, b.

\[ C = a \cap b \]
\[ D \in a \]
\[ c = \text{CIRCLE}(C, D) \]
\[ E = b \cap c \]
\[ M = \text{MIDPOINT}(D, E) \]
\[ m = \text{LINE}(C, M) \]

(9) **SQUARE**(A, B) is the pair of points C, D such that ABCD is a square.

\[ p = \text{PERPENDICULAR}(A, B) \]
\[ c = \text{CIRCLE}(B, A) \]
\[ C = p \cap c \]
\[ D = \text{PARALLELOGRAM}(A, B, C) \]

(10) **FOOT**(A, m) is the foot, F, of the perpendicular from A to m

\[ B \in m \]
\[ c = \text{CIRCLE}(A, B) \]
\[ B, C = m \cap c \]
\[ F = \text{MIDPOINT}(B, C) \]

**NOTE:** This fails if B happens to be the foot of the perpendicular already for then there is only one point of intersection of the circle with the line. In practice it is easy to choose a suitable point. But the precise formulation that we have given there is no place for this. Is there a modification to the above construction which will work in all circumstances? Yes, there is.

§9.3. **Some More Advanced Constructions**

Any circle that we construct in the course of a Ruler and Compass Construction will already have its centre identified. But suppose we are given a circle at the outset. How do we find its centre?

We simply take three distinct points P, P', P'' on the circle and join P to P' and P' to P. The perpendicular bisectors of PP' and PP'' intersect at the centre.

(11) **CENTRE**(c) is the centre, C, of the circle c

\[ P, P', P'' \in c \]
\[ b = \text{BISECT}(P, P') \]
\[ b' = \text{BISECT}(P, P'') \]
\[ C = b' \cap b'' \]
If we’re given a circle and a point on it, how do we construct the tangent at that point? We construct the centre, join the point to the centre and then draw the line perpendicular to this line.

(12) TANGENT(c, P) is the tangent, t, to the circle c at a point P lying on the circle.

C = CENTRE(c)
t = PERPENDICULAR(C, P)

Now we could be given other curves at the outset, curves that cannot be constructed. For example we might be given a parabola and be asked to construct its axis. Here we need to have some knowledge of the geometry of the parabola such as is normally taught in high-school coordinate geometry.

You may recall that the midpoints of parallel chords of a parabola lie on a line that is parallel to the axis. This leads to the following construction.

(13) AXIS(p) is the axis, x, of the parabola p

A, B, C ∈ p
a = LINE(B, C)
b = PARALLEL(A, a)
D = b ∩ p
E = MIDPOINT(A, D)
F = MIDPOINT(B, C)
c = LINE(F, G)
d = PERPENDICULAR(F, G)
e = PERPENDICULAR(G, F)
G, H = e ∩ p
M = MIDPOINT(H, K)
S, T = d ∩ p
N = MIDPOINT(S, T)
x = LINE(M, N)

(14) VERTEX(p) is the vertex, V, of the parabola p

x = AXIS(p)
VERTEX(p) = V = x ∩ p
(15) TANGENT \((P, p)\) is the tangent, \(t\), to the parabola \(p\) at the point \(P\).

\[ V = \text{VERTEX}(p) \]
\[ P \in p \text{ (ensure that } P \neq V) \]
\[ h = \text{AXIS}(p) \]
\[ b = \text{PERPENDICULAR}(P, h) \]
\[ Q = a \cap b \]
\[ R = \text{REFLECT}(Q, P) \]
\[ d = \text{PERPENDICULAR}(R, b) \]
\[ S = d \cap p \]
\[ c = \text{LINE}(V, S) \]
\[ t = \text{PARALLEL}(P, c) \]

[For the parabola \(x^2 = 4ay\), take \(P = (2at, at^2)\) where \(t \neq 0\).

The axis, \(h\), is \(x = 0\) and the line, \(b\), is \(y = at^2\). \(Q\) is \((0, at^2)\) and \(R\) is \((4at, at^2)\). The line \(d\) is \(x = 4at\) so \(S\) is \((4at, 4at^2)\). The slope of \(c\) is \(t\), which is the slope of the tangent.]

§9.4. Constructible Numbers

We identify the complex number with its corresponding point on the complex plane.

A **constructible number** is a complex number whose corresponding point on the complex plane can be constructed by ruler and compass, starting with two points representing 0 and 1.

**Theorem 1:** A complex number is constructible if and only if its real and imaginary parts are constructible.

**Proof:** The real and imaginary axes are \(a = \text{LINE}(0, 1)\) and \(b = \text{PERPENDICULAR}(1, 0)\) respectively.

Let \(Z = X + iY\). Then \(X = \text{FOOT}(Z, a)\) and \(iY = \text{FOOT}(Z, b)\).

Hence \(Y = \text{CIRCLE}(0, iY) \cap a\) so if \(Z\) is constructible then so are \(\text{Re}(Z)\) and \(\text{Im}(Z)\).

Conversely suppose \(X\) and \(Y\) are constructible. Then \(iY = \text{CIRCLE}(0, Y) \cap b\) and so \(Z = \text{PARALLELOGRAM}(X, 0, iY)\).

**Theorem 2:** The set of constructible numbers is a field.

**Proof:** Firstly the set of all real constructible numbers is a field since:

\(X + Y = \text{PARELLOGRAM}(X, 0, Y)\);
\(-X = \text{REFLECT}(X, 0)\)

Let \(X = e^{i\alpha}\) and \(Y = e^{i\beta}\).

\(a = \text{LINE}(0, Y)\)

\(F = \text{FOOT}(X, a)\)

\(XY = \text{REFLECT}(X, F)\)

represents \(XY = e^{i(\alpha+\beta)}\).
If $Y$ is a complex number and $X$ is real we can construct $XY$ as follows:

- $a = \text{LINE}(0, 1)$
- $b = \text{PERPENDICULAR}(1, 0)$
- $c = \text{CIRCLE}(0, Y)$
- $D = b \cap c$
- $E = \text{PARELLOGRAM}(D, 0, 1)$
- $d = \text{LINE}(0, E)$
- $e = \text{PERPENDICULAR}(0, X)$
- $F = d \cap e$
- $G = \text{PARELLOGRAM}(0, X, F)$
- $f = \text{CIRCLE}(0, G)$
- $g = \text{LINE}(0, Y)$
- $Z = f \cap g$

Multiplication of two complex numbers $X$ and $Y$ can be achieved by first multiplying $X$ by $|Y|$ and then multiplying by $\frac{Y}{|Y|}$ (except when $Y = 0$, but the constructability is obvious in this case). Since each of these takes constructible numbers to constructible numbers the constructability of $XY$ follows. Of course we must also check that $|Y|$ and $\frac{Y}{|Y|}$ are constructible if $Y$ is, but this is easy.

We also leave it as an exercise to show that if $X$ is a non-zero constructible complex number, $\frac{1}{X}$ is constructible.

**Corollary:** Rational numbers are constructible.
**Theorem 3:** The square roots of a constructible number are constructible.

**Proof:** The theorem is true for positive real constructible numbers as the following construction shows (we leave it as an exercise to obtain an explicit construction).

\[
\frac{1 + x}{2} \quad \frac{1 - x}{2} \quad \sqrt{x}
\]

For a non-real constructible number, \( z = re^{i\theta} \) we construct \( r = \text{CIRCLE}(0, z) \cap \text{LINE}(0, 1) \). Then we construct \( \sqrt{r} \) as above.

The square roots of \( z \) are \( \text{BISECT}(\text{LINE}(0, 1), \text{LINE}(0, z) \cap \text{CIRCLE}(0, \sqrt{r})) \).

**Example 1:** \( \sqrt{3 + \sqrt{3 + \sqrt{7}}} + \frac{\sqrt{5}}{\sqrt{3 + \sqrt{2}}} i \) is constructible (though it would probably be a nightmare to obtain an explicit construction).

While there are infinitely many constructible numbers, of arbitrary complexity, most complex numbers are not constructible. In a later chapter we’ll prove that \( \sqrt{2} \) and \( e^{2\pi i/9} \) are not constructible, thereby proving the impossibility of doubling the cube and trisecting the angle \( 2\pi/3 \), by ruler and compass. Another famous non-constructible number is \( \pi \) whose non-constructability establishes the impossibility of squaring the circle.

**EXERCISES FOR CHAPTER 9**

**Exercise 1:** Describe a ruler and compass construction to draw the chord of contact of tangents from the point \( P \), lying outside the circle \( c \).

**Exercise 2:** Find ruler and compass constructions for the inscribed circle for a triangle \( ABC \).

**Exercise 3:** Find a ruler and compass construction to construct the common tangents (where they exist) to two given circles of different radii.

**Exercise 4:** Show that \( \cos(2\pi/5) \) is constructible.

HINT: Use De Moivre’s Theorem to find a polynomial with integer coefficients having \( \cos(2\pi/5) \) as a root and hence express \( \cos(2\pi/5) \) as a surd expression involving square roots.

**Exercise 5:** Show that if the non-zero complex number \( Z \) is constructible then so is \( 1/Z \).
SOLUTIONS FOR CHAPTER 9

Exercise 1:
The tangents will be perpendicular to the radius. Since the angle in a semicircle is a right angle, we can construct a circle whose diameter is the line segment joining P to the centre of the circle. Where this cuts c are the points of contact.

Exercise 2:
The centre of the inscribed circle is the common intersection of the perpendicular bisectors of the sides.

Exercise 3:
(d) Let the circles be CIRCLE(C_1, P_1) of radius r_1 and CIRCLE(C_2, P_2) of radius r_2 where r_1 > r_2.

Construct CIRCLE(C_1, R) so that it has radius r_1 - r_2.

Construct M, the midpoint of C_1, C_2.

E_1, E_2 = CIRCLE(M, C_1) \cap CIRCLE(C_1, R).

T_1, T_2 = LINE(E_1, C_1) \cap CIRCLE(C_1, P_1).

Construct the line LINE(T_1, X) perpendicular to LINE(C_1, E_1).

This is one of the four common tangents.
Exercise 4: If \( c = \cos(2\pi/5) \) and \( s = \sin(2\pi/5) \) then by De Moivre’s Theorem,
\[(c + is)^5 = \cos(2\pi) + is\sin(2\pi) = 1.\]
Hence \( c^5 + 5isc^4 - 10s^2c^3 - 10is^3c^2 + 5s^4c + is^5.\)
Equating imaginary parts, and dividing by \( s \) we get \( 5c^4 - 10s^2c^2 + s^4 = 0.\)
So \( 5c^4 - 10(1 - c^2)c^2 + (1 - c^2)^2 = 0.\)
Hence \( 16c^4 - 12c^2 + 1 = 0.\)

So \( c^2 = \frac{12 \pm \sqrt{80}}{32} \) so \( c = \pm \sqrt[4]{\frac{12 \pm \sqrt{80}}{32}}. \) By eliminating three of these four alternatives we see that in fact \( c = -\sqrt[4]{\frac{12 - \sqrt{80}}{32}}. \) This is clearly constructible by ruler and compass.

Exercise 5:
\( a = \text{LINE}(0, 1) \)  \[\text{x-axis}\]
\( M = \text{CIRCLE}(0, Z) \cap a \)  \[\text{modulus of} \ Z\]
\( N = 1/M = 1/|Z| \) can be constructed as above
\( p = \text{PERPENDICULAR}(0, N) \)
\( U = p \cap \text{LINE}(0, Z) \)
\( I = \text{REFLECT} \ (U, N) \)  \[1/Z\]