§8.1. The Quadratic From An Advanced Standpoint

The formula for the solutions to a quadratic equation has been known since the time of the ancient Babylonians. Expressed in our modern day notation the solutions to \( ax^2 + bx + c = 0 \) are:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

The usual method of obtaining the quadratic formula involves a method called \textit{completing the square} which doesn’t generalize to higher degree polynomials. The following derivation of the quadratic formula gets more to the heart of the matter.

Let the roots of \( ax^2 + bx + c = 0 \) be \( \alpha \) and \( \beta \). Then it is well known that the sum of the roots and the product of the roots can be expressed very simply in terms of the coefficients:

\[
S = \alpha + \beta = -\frac{b}{a}
\]
\[
P = \alpha \beta = \frac{c}{a}
\]

Both the sum of the roots and the product of the roots are symmetric in terms of \( \alpha \) and \( \beta \). If \( \alpha \) and \( \beta \) are swapped they remain unchanged. These two functions of the roots are called the \textit{elementary symmetric functions} and other symmetric functions of the roots can be expressed in terms of them.

\textbf{Example 1}: Express each of the following symmetric functions in terms of the elementary symmetric ones:

(a) \( \alpha^2 \beta + \beta^2 \alpha \); (b) \( \frac{1}{\alpha} + \frac{1}{\beta} \); (c) \( \alpha^2 + \beta^2 \); (d) \( \alpha^3 + \beta^3 \).

\textbf{Solution}: 

(a) \( \alpha^2 \beta + \beta^2 \alpha = \alpha \beta (\alpha + \beta) = PS \);

(b) \(\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha \beta} = \frac{S}{P} \);

(c) \( \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2 \alpha \beta = S^2 - 2P \);

(d) \( \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3(\alpha^2 \beta + \alpha \beta^2) = S^3 - 3PS \).

It is obvious that any function which can be expressed in terms of \( \alpha + \beta \) and \( \alpha \beta \) must be symmetric in \( \alpha \) and \( \beta \). What is a little less well known is the fact that the converse holds. Every symmetric function in \( \alpha \) and \( \beta \) can be expressed in terms of \( \alpha + \beta \) and \( \alpha \beta \). It follows that the value of such functions can be computed directly from the coefficients without having to solve the quadratic.

Now an expression such as \( \alpha - \beta \) is not symmetric. Swapping \( \alpha \) and \( \beta \) in fact changes the sign of the expression. However if we square \( \alpha - \beta \), this change of sign disappears and we again get the symmetric function

\[
(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2 \alpha \beta = (\alpha + \beta)^2 - 4 \alpha \beta = S^2 - 4P
\]

\[
= \left( \frac{-b}{a} \right)^2 - 4 \left( \frac{c}{a} \right) = \frac{b^2 - 4ac}{a^2}
\]
Hence we can find the values of $\alpha - \beta$ simply by taking square roots, getting
\[
\alpha - \beta = \frac{\pm \sqrt{b^2 - 4ac}}{a}
\]
Now $\alpha + \beta = -\frac{b}{a}$ and so adding these equations and dividing by 2 we get
\[
\alpha = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
This formula can be expressed as an algorithm:
(1) Find $\Delta = b^2 - 4ac$;
(2) Solve $z^2 = \Delta$;
(3) Solve $2ax = z - b$.
Each step involves solving a linear equation or finding an $n$’th root.

§8.2. The Cubic Equation

The general cubic equation has the form $ax^3 + bx^2 + cx + d = 0$ where $a \neq 0$.
Let the zeros be $\alpha$, $\beta$, and $\gamma$. Then the elementary symmetric functions of these roots can be expressed directly in terms of the coefficients as follows:
\[
S = \alpha + \beta + \gamma = -\frac{b}{a}
\]
\[
Q = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}
\]
\[
P = \alpha\beta\gamma = -\frac{d}{a}
\]
As before, any symmetric function can be expressed in terms of these. For example:
\[
\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = S^2 - 2Q.
\]

Example 2: Express the following symmetric functions in terms of $P$, $Q$, $S$:
(a) $\alpha^2 + \beta^2 + \gamma^2$;  
(b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$;  
(c) $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$;  
(d) $\alpha^3 + \beta^3 + \gamma^3$.

Solution:
(a) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = S^2 - 2Q$;
(b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{Q}{P}$;
(c) $\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2 = (\alpha\beta + \alpha\gamma + \beta\gamma)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma = QS - 3P$;
(d) $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2) - 6\alpha\beta\gamma = S^3 - 3(QS - 3P) - 6P = S^3 - 3QS + 3P$. 

74
Other expressions have partial symmetry. For example consider:

\[ \Delta_1 = \alpha^2 \beta + \beta^2 \gamma + \gamma^2 \alpha \]
\[ \Delta_2 = \alpha \beta^2 + \beta \gamma^2 + \gamma \alpha^2. \]

These are not completely symmetric because under the permutation \((\alpha \beta)\) the expressions \(\Delta_1\) and \(\Delta_2\) change into one another. But \(\Delta_1\) and \(\Delta_2\) are symmetric under the permutations \((\alpha \beta \gamma)\) and its inverse \((\alpha \gamma \beta)\). Including the identity permutation which keeps all of \(\alpha, \beta\) and \(\gamma\) fixed we find that \(\Delta_1\) and \(\Delta_2\) are unchanged by three of the 6 permutations but are swapped by the other three. We could say that they have half-symmetry.

Other expressions have this half-symmetry. One notable example is the discriminant:

\[ \Delta = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \]
which can be written as \(\Delta_2 - \Delta_1\). For three of the 6 permutations on \{\(\alpha, \beta, \gamma\}\} \(\Delta\) is left fixed and for the other three \(\Delta\) is sent to \(-\Delta\).

This is just what we had with the quadratic discriminant. If we now square the discriminant we get something that is fully symmetric. And being fully symmetric we can express \(\Delta^2\) in terms of the elementary symmetric functions \(S, Q\) and \(P\) and hence we can find \(\Delta^2\) in terms of the coefficients. All we have to do is to take the square root and we have found \(\Delta\).

\[
\begin{align*}
\Delta_1 &= \alpha^2 \beta + \beta^2 \gamma + \gamma^2 \alpha; \\
\Delta_2 &= \alpha \beta^2 + \beta \gamma^2 + \gamma \alpha^2; \\
\Delta_1 + \Delta_2 &= (\alpha \beta + \beta \gamma + \gamma \alpha)(\alpha + \beta + \gamma) - 3\alpha \beta \gamma \\
&= QS - 3P; \\
\Delta_1 \Delta_2 &= PS^3 + Q^3 - 6SPQ + 9P^2; \\
(\Delta_1 - \Delta_2)^2 &= (\Delta_1 + \Delta_2)^2 - 4\Delta_1 \Delta_2 \\
&= (QS - 3P)^2 - 4(PS^3 + Q^3 - 6SPQ + 9P^2) \\
&= Q^2 S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3.
\end{align*}
\]

From these equations we can find \(\Delta_1\) and \(\Delta_2\).

**Example 3:** Find \(\Delta_1\) and \(\Delta_2\) for the polynomial \(x^3 - 3x - 2\).

**Solution:** \(S = 0, Q = -3, P = 2\).

\[
\begin{align*}
\Delta_1 + \Delta_2 &= QS - 3P = -6; \\
(\Delta_1 - \Delta_2)^2 &= Q^2 S^2 - 27P^2 - 4PS^3 + 18SPQ - 4Q^3 = -108 + 108 = 0
\end{align*}
\]
Hence \(\Delta_1 - \Delta_2 = 0\) and so \(\Delta_1 = \Delta_2 = -3\).

For the quadratic equation, the role of \(\Delta_1\) and \(\Delta_2\) was played by the roots \(\alpha\) and \(\beta\) themselves. But with the cubic, we have a bit more work to do.

Let \(E = \alpha + \beta \omega + \gamma \omega^2\)
and \(F = \alpha + \beta \omega^2 + \gamma \omega\).

These are not even half-symmetric because the cycle \((\alpha \beta \gamma)\) changes \(E\) to \(\omega^2 E\) and changes \(F\) to \(F \omega\). But if we cube \(E\) and \(F\) then the \(\omega\) and \(\omega^2\) will disappear. This means that \(E^3\) and \(F^3\) are half-symmetric. Maybe we can express them in terms of \(S,\)
Q, P plus the half-symmetric expressions $\Delta_1$ and $\Delta_2$. If so then we can take cube roots to find the values of E and F.

Now in fact: $E^3 = (\alpha + \beta \omega + \gamma \omega^2)^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$ and $F^3 = (\alpha + \beta \omega^2 + \gamma \omega)^3 = S^3 - 3QS + 9P + 3\omega\Delta_1 + 3\omega^2\Delta_2$.

**Example 4:** Find E and F for the polynomial $x^3 - 3x - 2$.

**Solution:** In example 3 we found that $S = 0$, $Q = -3$, $P = 2$ and $\Delta_1 = \Delta_2 = -3$.

$E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2$

$= 0 - 0 + 18 -9\omega^2 -9\omega$

$= 27 - 9(1 + \omega + \omega^2) = 27$

and similarly $F^3 = 27$.

Thus we have three possibilities for each of E and F:

$E, F = 3, 3\omega$ or $3\omega^2$.

There are 9 combinations of these values but not all of them will produce solutions because $EF = (\alpha + \beta \omega + \gamma \omega^2)(\alpha + \beta \omega^2 + \gamma \omega)$

$= \alpha^2 + \beta^2 + \gamma^2 + (\alpha \beta + \beta \gamma + \gamma \alpha)(\omega + \omega^2)$

$= S^2 - 2Q - Q$

$= S^2 - 3Q$.

So provided $E \neq 0$ we can find F as $E^3 - 3Q$. If $E = 0$ we must find $F^3$ and take cube roots.

So in fact we only have 3 possibilities for E, F. It is easy to show that any one will do. The other cases simply give the zeros in different orders.

Now having values for E and F, how do we get our hands on the zeros $\alpha$, $\beta$ and $\gamma$ themselves? That's easy! We have:

$\alpha + \beta + \gamma = S$

$\alpha + \beta \omega + \gamma \omega^2 = E$

$\alpha + \beta \omega^2 + \gamma \omega = F$.

If we simply add these equations, and use the relationship $1 + \omega + \omega^2 = 0$, we get

$3\alpha = S + E + F$ and so $\alpha = \frac{S + E + F}{3}$.

Similarly $\beta = \frac{S + E\omega^2 + F\omega}{3}$ and $\gamma = \frac{S + E\omega + F\omega^2}{3}$.

**Example 5:** Solve the cubic $x^3 - 3x - 2 = 0$.

**Solution:** Remember that $S = 0$, so $\alpha = \frac{S + E + F}{3}$.

Taking $E = F = 3$ we get $\alpha = 2$, $\beta = \gamma = -1$.

We can describe the process of solving a cubic in the following table. The second and third columns give two examples.
CUBIC EQUATION

### GENERAL CASE

<table>
<thead>
<tr>
<th>(ax^3 + bx^2 + cx + d)</th>
<th>EXAMPLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^3 - 4x^2 + 4x - 3)</td>
<td>(x^3 - 6x^2 + 12x + 9)</td>
</tr>
<tr>
<td>(S = -b/a)</td>
<td>4</td>
</tr>
<tr>
<td>(Q = c/a)</td>
<td>4</td>
</tr>
<tr>
<td>(P = -d/a)</td>
<td>3</td>
</tr>
<tr>
<td>(\Delta_1 + \Delta_2 = QS - 3P)</td>
<td>7</td>
</tr>
<tr>
<td>((\Delta_1 - \Delta_2)^2 = Q^2S^2 - 27P^2 - 4PS^3 + 18SQ - 4Q^3)</td>
<td>-147</td>
</tr>
<tr>
<td>(\Delta_1 - \Delta_2 = \text{either square root of the above})</td>
<td>7(\sqrt{3}i)</td>
</tr>
<tr>
<td>(\Delta_1 = (\Delta_1 + \Delta_2) + (\Delta_1 - \Delta_2) \over 2)</td>
<td>(7(1 + \sqrt{3}i) \over 2)</td>
</tr>
<tr>
<td>(\Delta_2 = (\Delta_1 + \Delta_2) - \Delta_1)</td>
<td>(7(1 - \sqrt{3}i) \over 2)</td>
</tr>
<tr>
<td>(E^3 = S^3 - 3QS + 9P + 3\omega^2 \Delta_1 + 3\omega \Delta_2)</td>
<td>64</td>
</tr>
<tr>
<td>(F^3 = S^3 - 3QS + 9P + 3\omega \Delta_1 + 3\omega^2 \Delta_2)</td>
<td>1</td>
</tr>
<tr>
<td>(EF = S^2 - 3Q)</td>
<td>4</td>
</tr>
<tr>
<td>(E = \text{any cube root of} \ E^3)</td>
<td>4</td>
</tr>
<tr>
<td>(F = (EF)/E \text{ or any cube roots of} \ F^3)</td>
<td>1</td>
</tr>
<tr>
<td>(\alpha = S + E + F \over 3)</td>
<td>3</td>
</tr>
<tr>
<td>(\beta = S + E\omega + F\omega^2 \over 3)</td>
<td>(1 + \omega)</td>
</tr>
<tr>
<td>(\gamma = S - (\alpha + \beta))</td>
<td>(-\omega)</td>
</tr>
</tbody>
</table>

**NOTE:** The cube roots must match so that \(EF = S^2 - 3Q\).

So if \(E \neq 0\) calculate \(F\) as \((EF)/E\). (In this case there is no need to calculate \(F^3\).)

However if \(E = 0\) then \(F\) must be calculated as a cube root of \(F^3\). It is possible to summarize the whole process into a single formula as follows.

Firstly, in order to keep the formula simple, we divide through by the coefficient of \(x^3\) to get the cubic in the form \(x^3 - Sx^2 + Qx - P\), so that \(S\) is the sum of the roots etc. Then, observing that the transformation \(y = x - S/3\) leads to a cubic with no \(x^2\) term, we can (without loss of generality) consider cubic equations of the form:

\[x^3 + Qx - P = 0\]

The zeros can be expressed by the formula:

\[\alpha, \beta, \gamma = \sqrt[3]{\frac{p}{2} + \sqrt{\frac{p^2}{4} + \frac{Q^3}{27}}} + \sqrt[3]{\frac{p}{2} - \sqrt{\frac{p^2}{4} + \frac{Q^3}{27}}}
\]

The cube roots are computed over the complex field and are chosen so that the product of these terms is \(-Q/3\).

**Example 6:** Solve \(x^3 - 6x - 6 = 0\).

**Solution:** \(P = 6, Q = -6\) so one solution is \(\sqrt[3]{4} + \sqrt[3]{2}\). (This is the only real solution.)
EXERCISES FOR CHAPTER 8

Exercise 1:
Suppose $\alpha, \beta$ are the zeros of a quadratic $ax^2 + bx + c$ and let $S = \alpha + \beta$, $P = \alpha \beta$.
(i) Show that $\alpha^5 + \beta^5 = S^5 - 5PS^3 + 5P^2S$.
(ii) Express $K = \frac{\alpha}{\alpha^2 + \beta^2} + \frac{\beta}{\beta^2 + \alpha^2}$ in terms of $S$ and $P$.

Exercise 2:
Suppose $\alpha, \beta, \gamma$ are the zeros of a cubic and let $S = \alpha + \beta + \gamma$, $Q = \alpha \beta + \alpha \gamma + \beta \gamma$, $P = \alpha \beta \gamma$.
(i) Express $G = \alpha^2 \beta + \alpha^2 \gamma + \beta^2 \alpha + \beta^2 \gamma + \gamma^2 \alpha + \gamma^2 \beta$ in terms of $S$, $Q$ and $P$.
(ii) Express $H = \alpha^5 \beta^3 \gamma + \alpha^5 \beta \gamma^3 + \beta^5 \alpha^3 \gamma + \beta^5 \alpha \gamma^3 + \alpha \gamma^5 \beta + \gamma^5 \beta^3 \alpha$ in terms of $S$, $Q$ and $P$.

Exercise 3:
Let $G$ be the group of all permutations on $\{\alpha, \beta, \gamma, \delta\}$.
(i) Show that $E = \alpha \beta + \gamma \delta$ is fixed by a subgroup of $G$ which is isomorphic to $D_8$.
(ii) Let $F = \alpha + \beta i - \gamma - \delta i$. Prove that $F^4$ is fixed by the permutation $(\alpha \beta \gamma \delta)$.

Exercise 4:
(i) Solve the cubic $x^3 - 4x^2 + 4x - 3$ by computing $S$, $Q$, $P$, $\Delta_1$, $\Delta_2$, $E$, $F$.
(ii) Solve the cubic $x^3 - 6x^2 + 12x + 3$ by computing $S$, $Q$, $P$, $\Delta_1$, $\Delta_2$, $E$, $F$.

SOLUTIONS FOR CHAPTER 8

Exercise 1: (i) $\Sigma \alpha^5 = (\Sigma \alpha)^5 - 5.\Sigma \alpha^4 \beta - 10.\Sigma \alpha^3 \beta^2 = S^5 - 5P.\Sigma \alpha^3 - 10P^2S$
Now $\Sigma \alpha^3 = (\Sigma \alpha)^3 - 3.\Sigma \alpha^2 \beta = S^3 - 3PS$.
Thus $\Sigma \alpha^5 = S^5 - 5P(S^3 - 3PS) - 10P^2S = S^5 - 5PS^3 + 5P^2S$.
(ii) $E = \frac{\Sigma \alpha^3 \beta + \Sigma \alpha^3}{\alpha^2 \beta^2 + \alpha^2 \beta^2 + \Sigma \alpha^3} = \frac{P(S^2 - 2P) + S^3 - 3PS}{P^2 + P^2 + (S^2 - 5PS^3 + 5P^2S)}$
$= \frac{PS^2 - 2P^2 + S^3 - 3PS}{P^2 + P^2 + S^2 - 5PS^3 + 5P^2S}$

Exercise 2: (i) $G = \Sigma \alpha^2 \beta = \Sigma \alpha \beta$ . $\Sigma \alpha - 3P = QS - 3P$
(ii) $F = P\Sigma \alpha^2 \beta^2 = P[(\Sigma \alpha^2 \beta)^2 - 2\Sigma \alpha^2 \beta \gamma - 2\Sigma \alpha^2 \beta^3 - 2\Sigma \alpha^3 \beta^2 \gamma - 6P^2]$
$= PG^2 - 2P^2S^2 - 2P^2G - 6P^3$
Now $\Sigma \alpha^3 = (\Sigma \alpha)^3 - 3\Sigma \alpha^2 \beta - 6P = S^3 - 3G - 6P$ and $\Sigma \alpha^3 \beta^3 = (\Sigma \alpha \beta)^3 - 3\Sigma \alpha^2 \beta \gamma - 6P^2 = Q^3 - 3PG - 6P^2$.
Thus $H = PG^2 - 2P^2(S^3 - 3G - 6P) - 2P(Q^3 - 3PG - 6P^2) - 2P^2G - 6P^3$
$= PG^2 - 2P^2S^3 + 6P^2G + 12P^3 - 2PQ^3 + 6P^2G + 12 P^3 - 2P^2G - 6P^3$
$= PG^2 - 2P^2S^3 + 10P^2G + 18P^3 - 2PQ^3$
$= 18P^3 - 2PQ^3 - 2P^2S^3 + 10P^2(QS - 3P) + P(QS - 3P)^2$
$= 18P^3 - 2PQ^3 - 2P^2S^3 + 10P^2QS - 30P^3 + P(Q^2S^2 + 9P^2 - 6PQS)$
$= -3P^3 - 2PQ^3 - 2P^2S^3 + 4P^2QS + PQ^2S^2$. 

78
Exercise 3: (i) E is fixed by \{I, (\alpha\beta), (\gamma\delta), (\alpha\gamma\beta\delta), (\alpha\beta)(\gamma\delta), (\alpha\delta\beta\gamma), (\alpha\gamma)(\beta\delta), (\alpha\delta)(\beta\gamma)\}. If A = (\alpha\gamma\beta\delta) and B = (\alpha\beta) then G = \langle A, B \mid A^4 = B^2 = 1, BA = A^{-1}B \rangle 
(ii) F = \alpha + \beta i - \gamma - \delta i \rightarrow \beta + \gamma i - \delta - \alpha i = -i(\alpha + \beta i - \gamma - \delta i) = -iF so F^4 \to F^4.

Exercise 4:
(i) S = Q = 4, P = 3. \Delta_1 + \Delta_2 = SQ - 3P = 7, (\Delta_1 - \Delta_2)^2 = S^2 - 27P^2 + 18SPQ - 4Q^3 - 4PS^3 = -147 so let \Delta_1 - \Delta_2 = \sqrt{147} i = 7\sqrt{3} i. Hence \Delta_1 = \frac{7 + 7\sqrt{3} i}{2} = -7\omega^2 and \Delta_2 = -7\omega.
E^3 = S^3 - 3QS + 9P + 3\omega^2\Delta_1 + 3\omega\Delta_2 = 43 - 21\omega - 21\omega^2 = 43 + 21 = 64 so take E = 4.
EF = S^2 - 3Q = 4 so F = 1.
The zeros are thus \frac{4 + 4 + 1}{3} = 3, \frac{4 + 4\omega + \omega^2}{3} = 1 + \omega = -\omega^2 and \frac{4 + 4\omega^2 + \omega}{3} = 1 + \omega^2 = -\omega.
(ii) S = 6, Q = 12, P = 9. \Delta_1 + \Delta_2 = 45, (\Delta_1 - \Delta_2)^2 = -27 so take \Delta_1 - \Delta_2 = 3\sqrt{3} i.
Hence \Delta_1 = \frac{45 + 3\sqrt{3} i}{2} and \Delta_2 = \frac{45 - 3\sqrt{3} i}{2}.
E^3 = 0 so E = 0. F^3 = -3 so take F = -3.
The zeros are thus 3, 2 + \omega and 2 + \omega^2.