12. DIFFERENTIAL EQUATIONS

12.1 What is a Differential Equation?

Frequently analysis of a real-world system shows that there is some relationship between the value of a function and the value of its derivatives. Such a relationship is called a differential equation. A first order DE (short for “differential equation”) involves just the first derivative. A second order DE involves the second derivative as well. We won’t go beyond second order DEs. In fact, we’ll only scratch the surface of the very deep subject of differential equations.

Example 1: Solve the differential equation \( \frac{dy}{dx} = 100x^3 \).

Solution: Differentiate a function and you get 100x^3. What was the original function? This is simply asking us to integrate y with respect to x. The solution is \( y = 25x^4 + c \).

Example 2: Solve the differential equation \( \frac{d^2y}{dx^2} = 100x^3 \).

Solution: Differentiate twice and you get 100x^3. What did you start with? This requires us to integrate twice.

\[
\frac{d^2y}{dx^2} = 100x^3 \quad \text{so} \quad \frac{dy}{dx} = 25x^4 + a \quad \text{for some arbitrary constant “a”}.
\]

Integrating again we get \( y = 5x^5 + ax + b \).

Here we get arbitrary constants. We have to use different symbols. Moreover the first arbitrary constant got integrated into ax. So an arbitrary constant “c” doesn’t always show up as “+ c” in the final answer.

You’ll notice that with our first order DE we got one arbitrary constant and with our second order DE we got two. This is usually the case for those DEs that arise in real-world situations.

Example 3: Solve \( x^2 \frac{d^2y}{dx^2} = 1 \).

Solution: We rewrite this as \( \frac{d^2y}{dx^2} = \frac{1}{x^2} = x^{-2} \).

So \( \frac{dy}{dx} = -\frac{1}{x} + a \) and hence \( y = -\log x + ax + b \).

This type of differential equation is very easy to solve (as long as we can do the necessary integrations). These DEs have the form \( \frac{d^n y}{dx^n} = f(x) \) and we simply integrate \( n \) times, picking up arbitrary constants as we go. We now look at a different type of Differential equation.
12.2 Exponential Growth and Decay

A differential equation that crops up in many applications is:

\[ \frac{dx}{dt} = kx. \]

The equation says that the rate of increase of \( x \) is proportional to \( x \). There are numerous situations where this is the case. Here we use \( t \) as the independent variable because in most applications the independent variable is time.

(1) Investment

If you invest money at a fixed rate of compound interest, then the rate at which your money grows is proportional to the amount you have invested. Once your money has doubled so does your interest, even with the same rate of interest.

If the annual rate of interest is \( r\% \) then the balance of your account \( B \) will obey the differential equation \( \frac{dB}{dt} = \frac{r}{100}B. \) Here the value of \( k \) is \( \frac{r}{100} \).

(2) Population Growth

During a time when birth and death rates are constant, and there is no net immigration or migration, the population of a country, \( P \), will grow subject to the differential equation \( \frac{dP}{dt} = kP \) where \( k \) is the difference between the birth rates and death rates, expressed as a fraction. If the birth rate is 3\% per year and the death rate is 2\% then the net rate of increase would be 1\% and so \( k = 0.01 \).

(3) Radioactive Decay

The amount of radioactive substance that’s lost, through particles being emitted, is proportional to quantity remaining. In this case, because the amount is decreasing, the value of \( k \) is negative. For example, if a block of radioactive substance decays at the rate of 5\% per year then the amount, \( A \), obeys the equation \( \frac{dA}{dt} = -0.05A. \)

(4) Newton’s Law of Cooling

You’ll have noticed that a cup of hot tea cools down very quickly at first. But it can remain lukewarm, a few degrees above room temperature, for quite a long time. One of the many things that Isaac Newton discovered was that when an object cools, the rate of decrease of temperature is proportional to the difference between its current temperature and room temperature.

If \( T \) is the number of degrees above the “ambient temperature” (the temperature of the surroundings) then \( \frac{dT}{dt} = -kT. \) By solving this equation we can predict what the temperature will be after a certain length of time.

When a man dies his body cools. We have a pretty fair idea of his body temperature at death and if the room temperature has been more or less constant the temperature when the body is found can give us an estimate of when he died. This is because the temperature difference (body temperature – room temperature) obeys the growth and decay differential equation.
12.3 The Solution to the Growth and Decay Equation

In the first section we wrote the independent variable as “x” and the dependent variable as “y” and so the equations involved \( \frac{dy}{dx} \). But usually, in applications of the Growth and Decay Equation, the independent variable is “t” for time. So in this section we’ll write the Growth and Decay equation in the form \( \frac{dx}{dt} = kx \). What is its solution?

The trick to solving it is to turn the equation upside down to get \( \frac{dt}{dx} = \frac{1}{kx} \). This is just like the equation \( \frac{dy}{dx} = \frac{1}{kx} \) but with different variable names.

So if \( \frac{dt}{dx} = \frac{1}{kx} \) we simply need to integrate “t” with respect to “x” to get

\[
t = \int \frac{1}{kx} \, dx = \frac{1}{k} \int \frac{1}{x} \, dx = \frac{1}{k} \log x + c.
\]

Multiplying by \( k \) we get:

\[
\log x = kt - kc.
\]

Since “logs are powers”, \( kt - kc \) is the power that “e” must be raised to in order to equal \( x \).

Hence \( x = e^{kt-kc} = e^{-kc}e^{kt} \)

Now \( k \) is a constant and \( c \) is an arbitrary constant and so \( e^{-kc} \) is a more or less arbitrary constant. It’s not completely arbitrary because it has to be positive. But it can take any positive value. So let’s put \( A = e^{-kc} \).

So we now have, as our solution, \( x = Ae^{kt} \).

The general solution to the DE

\[
\frac{dx}{dt} = kx
\]

is \( x = Ae^{kt} \) where \( A \) is an arbitrary constant.

12.4 Some Second Order Differential Equations

We conclude our discussion of differential equations with another special type, those of the form:

\[
a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = 0 \text{ where } a, b, c \text{ are constants.}
\]

If \( a = 0 \) this becomes a first-order DE and reduces to \( b \frac{dx}{dt} + cx = 0 \). If \( b \neq 0 \) we can write it as \( \frac{dx}{dt} = \left( -\frac{c}{b} \right) x \), which is one of the Growth and Decay type, with the exponential solution we found in the last section.

So let’s suppose that \( a \neq 0 \) and we have a genuine second order DE of this type. Inspired by the first-order case, let’s see if we have an exponential solution \( x = e^{\lambda t} \) for some \( \lambda \).

If \( x = e^{\lambda t} \) then \( \frac{dx}{dt} = \lambda e^{\lambda t} \) and \( \frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t} \). Substituting into the DE we get:

\[
a \lambda^2 e^{\lambda t} + b \lambda e^{\lambda t} + c e^{\lambda t} = 0
\]

and so \( (a \lambda^2 + b \lambda + c) e^{\lambda t} = 0 \).

Since \( e^{\lambda t} \) is always positive we’d need to have \( a \lambda^2 + b \lambda + c = 0 \).

So we could choose \( \lambda \) to be any solution of this quadratic equation, which we call the auxiliary equation of the DE.
Now there are three possibilities. The quadratic may have no real solutions (that’s if the discriminant \( b^2 - 4ac < 0 \)). Or it may have two equal, real solutions. Here \( b^2 - 4ac = 0 \). The DE has solutions in each of these cases, but since we’re giving just a glimpse here, we’ll only consider the case where \( b^2 - 4ac > 0 \). In this case the quadratic has two distinct real solutions.

If these are \( r, s \) then \( x = e^{rt} \) and \( x = e^{st} \) are solutions to the DE. But it’s easily seen that any combination of them of the form \( x = Ae^{rt} + Be^{st} \), where \( A, B \) are arbitrary constants, will also be a solution.

For if \( x = Ae^{rt} + Be^{st} \) then \( \frac{dx}{dt} = rAe^{rt} + sBe^{st} \) and

\[
\frac{d^2x}{dt^2} = r^2Ae^{rt} + s^2Be^{st}.
\]

Substituting into the left-hand side of the DE we get:

\[
a(r^2Ae^{rt} + s^2Be^{st}) + b(rAe^{rt} + sBe^{st}) + c(Ae^{rt} + Be^{st})
= (ar^2 + br + c)Ae^{rt} + (as^2 + bs + c)Be^{st} = 0 + 0 = 0
\]

What’s not clear here is that there are no other solutions. As circumstantial evidence let me offer the fact that we have two arbitrary constants, which is what one is supposed to have for a second-order DE. But we never actually proved that, and nor are we going to. Suffice to say, what we found is the general solution in this case.

**Example 5:** Solve \( \frac{d^2x}{dt^2} - 5 \frac{dx}{dt} + 6x = 0 \).

**Solution:** The auxiliary equation is \( \lambda^2 - 5\lambda + 6 = 0 \). The left-hand side can be factored as \( (\lambda - 2)(\lambda - 3) \) so the quadratic has solutions \( \lambda = 2, 3 \).

The general solution is therefore \( x = Ae^{-3t} + Be^{3t} \) for arbitrary constants \( A, B \).

### 12.5 Initial Conditions

The general solution to a DE contains arbitrary constants. But if we have additional information these constants are no longer arbitrary and we may be able to determine their values. When this information involves only one value of \( t \), we have **initial conditions**. Less commonly, the information can involve more than one value of \( t \) (e.g. \( x = 1 \) when \( t = 0 \) and \( x = 4 \) when \( t = 1 \)), in which case we have **boundary conditions**.

**Example 6:** Solve \( \frac{d^2x}{dt^2} - 9x = 0 \) if \( x = 4 \) and \( \frac{dx}{dt} = -6 \) when \( t = 0 \).

**Solution:** The auxiliary equation is \( \lambda^2 - 9 = 0 \) which has solutions \( \lambda = -3, 3 \).

So the general solution to the DE is \( x = Ae^{-3t} + Be^{3t} \).

Then \( \frac{dx}{dt} = -3Ae^{-3t} + 3Be^{3t} \).

Put \( t = 0 \) and \( x = 4 \). Then \( 4 = A + B \).

Put \( t = 0 \) and \( \frac{dx}{dt} = -6 \). Then \( -6 = -3A + 3B \) so \( A - B = 2 \).

We now have to solve the pair of simultaneous equations:

<table>
<thead>
<tr>
<th>Suppose the quadratic ( a\lambda^2 + b\lambda + c = 0 ) has distinct solutions ( r, s ), Then the general solution to the DE ( \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = 0 ) is ( x = Ae^{rt} + Be^{st} ).</th>
</tr>
</thead>
</table>

158
Adding we get $2A = 6$ so $A = 3$. Hence $B = 1$.

The solution is $x = 3e^{-3t} + e^{3t}$.

**EXERCISES FOR CHAPTER 12**

**Exercise 1:**
Solve the following differential equations, using the initial condition where it is provided:

(i) $\frac{dx}{dt} = 3x$.

(ii) $\frac{dy}{dx} + 2y = 0$.

(iii) $3\frac{dy}{dx} - y = 0$.

(iv) $\frac{dx}{dt} = 2x$ where $x = 10$ when $t = 0$.

(v) $\frac{du}{dx} = -x$ where $u = 7$ when $t = 0$.

(vi) $\frac{dy}{dx} = 3y$ where $y = 10$ when $x = 1$.

**Exercise 2:**
By making the substitution $u = y - 1$, solve the differential equation $\frac{dy}{dx} = y - 1$ where $y = 2$ when $x = 0$.

**Exercise 3:**
Suppose you invest $100,000 at an annual rate of 5%. Interest is added to the investment daily. The amount of your investment after $t$ years is $A$ where $A$ is a function of $t$.

[Because the interest is calculated daily you can consider the process to be a continuous one, and so use calculus.]

(a) Write down a differential equation of the form $\frac{dA}{dt} = kt$ which describes the growth of your investment.

(b) Solve this differential equation.

(c) Use you solution to calculate the amount of interest you will have earned after 3 years.

(d) Now suppose that your interest was paid at the end and had not been compounded. Work out the interest that would have been obtained under these circumstances.

**Exercise 4:**
The population of a colony of beetles is, at the moment, 50,000. The birth rate is 5% per week and the death rate is 4% per week.

(a) Estimate the population after one year.

(b) How long would it take for the population to double?

**Exercise 5:**
A certain radioactive material decays at the rate of 5% per year. How long will it take for the level of radioactivity to drop to half? (This is known as the half-life of the material.)
Exercise 6:
A body is discovered at 6 am one morning in his apartment. The temperature of the body was found to be 32°C. The room was air-conditioned at a constant 20°C. Assuming that the difference between body temperature and surrounding temperature drops 6% per hour, at approximately what time did the murder occur? (Normal body temperature is 38°C.)

Exercise 7: Solve the following differential equations, using any initial conditions that might be given:

(i)  \( \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0. \)

(ii)  \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 15y \) where \( y = 4 \) and \( \frac{dy}{dx} = 2 \) when \( x = 0. \)

(iii)  \( \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} = 0 \) where \( y = 3 \) and \( \frac{dy}{dx} = 7 \) when \( x = 0. \)

Exercise 8: Suppose that \( y \) is a solution of the differential equation \( \frac{d^2y}{dx^2} + 2y = 3 \frac{dy}{dx} \) and that \( y = 1 \) when \( x = 0 \) and \( y = 2 \) when \( x = 1. \) Show that the value of \( y \) when \( x = -1 \) is \( e^{2} + e^{-2} \frac{e^3}{e^3}. \) Evaluate this to four decimal places.

SOLUTIONS FOR CHAPTER 12

Exercise 1:
(i)  \( x = Ae^{3t} \) for some constant \( A. \)

(ii)  \( \frac{dy}{dx} = -2y \) so \( y = Ae^{-x} \) for some constant \( A. \)

(iii)  \( \frac{dy}{dx} = \frac{1}{5} y \) so \( y = Ae^{(1/5)x} \) for some constant \( A. \)

(iv)  \( x = Ae^{2t} \) for some constant \( A. \)

When \( t = 0 \) this gives \( 10 = Ae^{0} = A \) so \( A = 10. \)

Hence the solution is \( x = 10e^{2t}. \)

(v)  \( u = Ae^{-x} \) for some constant \( A. \)

When \( t = 0 \) this gives \( 7 = Ae^{0} = A \) so \( A = 7. \)

Hence the solution is \( u = 7e^{-x}. \)

(vi)  \( y = Ae^{3x} \) for some constant \( A. \)

When \( x = 1 \) this gives \( 10 = Ae^{3} \) so \( A = \frac{10}{e^{3}}. \)

Hence the solution is \( y = \frac{10}{e^{3}} e^{3x} = 10e^{3(x-1)}. \)

Exercise 2:
Let \( u = y - 1. \) Then \( \frac{du}{dx} = \frac{dy}{dx}. \) Hence \( \frac{du}{dx} = u \) so \( u = Ae^{x} \) for some constant \( A. \)

Hence \( y = u + 1 = Ae^{x} + 1. \)

When \( x = 0, \) \( y = 2 \) and so \( 2 = Ae^{0} + 1 = A + 1, \) which gives \( A = 1. \)

The solution is therefore \( y = e^{x} + 1. \)
Exercise 3:
(a) \( \frac{dA}{dt} = 0.05A \).

(b) \( A = Ke^{0.05t} \) for some constant \( K \).
But when \( t = 0 \), \( A = 100,000 \) so \( K = 100,000 \).
Hence \( A = 100,000e^{0.05t} \).

(c) When \( t = 3 \), \( A = 100,000e^{0.015} \approx 116183.42 \). So after 3 years the amount invested will be $116,183.42. The interest earned will therefore be $16,183.42.

(d) If the interest was calculated only at the end the interest would have been $100,000 \times 0.05 \times 3 = $15,000.

Exercise 4:
(a) Let the population in \( t \) weeks time be \( P \), as a function of \( t \).
Then \( \frac{dP}{dt} = (0.05 - 0.04)P = 0.01P \).
Thus \( P = 50,000e^{0.01t} \).
After one year (52 weeks) the population will be \( 50,000e^{0.052} \approx 84,100 \).

(b) We want to solve \( 100,000 = 50,000e^{0.01t} \). Thus \( e^{0.01t} = 2 \) and so \( 0.01t = \log 2 \) (taking logs of both sides). This gives \( t = \frac{\log 2}{0.01} = 100\log 2 \approx 69.3 \).
Hence it will take just over 69 weeks for the population to double.

Exercise 5:
Let the level of radioactivity be \( L \), as a function of \( t \) where \( t \) is time measured in years. (It doesn’t matter what the units of radioactivity are.)
Then \( \frac{dL}{dt} = -0.05L \) and hence \( L = L_0e^{-0.05t} \) where \( L_0 \) is the level at time \( t = 0 \).
When \( t \) is the half-life (in years), \( L = \frac{L_0}{2} \) so \( \frac{L_0}{2} \approx L_0e^{-0.05t} \).
Solving this we get \( e^{-0.05t} = \frac{1}{2} \).
\[ \therefore e^{0.05t} = 2. \]
\[ \therefore 0.05t = \log 2. \]
\[ \therefore t = \frac{\log 2}{0.05} = 20\log 2 \approx 13.86. \]
So the half-life of the material is about 13.9 years.

Exercise 6:
Let \( T \) be the difference between the body temperature and that of the surroundings (in degrees centigrade) \( t \) hours after the murder.
Then \( \frac{dT}{dt} = -0.06T \). \( \therefore T = Ae^{-0.06t} \) for some constant \( A \).
When \( t = 0 \), \( T = 38 - 20 = 18 \). Hence \( T = 18e^{-0.06t} \).
Now let \( t \) be the number of hours between the murder and the time the body was found.
Then \( 32 - 20 = 18e^{-0.06t} \).
\[ \therefore e^{-0.06t} = \frac{12}{18} = \frac{2}{3}. \]


\[ \therefore e^{0.06t} = \frac{3}{2}. \]

\[ \therefore 0.06t = \log(3/2). \]

\[ \therefore t = \frac{\log(3/2)}{0.06} \approx 6.7577. \]

So the murder was committed about 6¾ hours before it was discovered, that is, at approximately **11:45 pm the night before**.

**Exercise 7:**

(i) The auxiliary equation is \( \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4) = 0 \) when \( \lambda = 3, 4 \).

The solution to the differential equation is thus \( y = Ae^{3x} + Be^{4x} \) for arbitrary constants A, B.

(ii) \[ \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 15y \] when \( y = 4 \) and \( \frac{dy}{dx} = 2 \) when \( x = 0 \).

The auxiliary equation is \( \lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3) = 0 \) when \( \lambda = -5, 3 \).

The solution to the differential equation is thus \( y = Ae^{-5x} + Be^{3x} \) for constants A, B.

When \( x = 0 \), \( y = 4 \) so \( 4 = A + B \).

Now \( \frac{dy}{dx} = -5Ae^{-5x} + 3Be^{3x} \).

When \( x = 0 \), \( \frac{dy}{dx} = 2 \) so \(-5A + 3B = 2 \).

So we must solve the following system of equations:

\[
\begin{align*}
A + B &= 4 \\
-5A + 3B &= 2 \\

\end{align*}
\]

Multiplying the first equation by 3 and subtracting the second we get \( 3A + 5A = 12 - 2 \).

\[ \therefore 8A = 10 \] and so \( A = \frac{5}{4} \).

Substituting into the first equation gives \( B = \frac{11}{4} \).

Hence the solution is \( y = \frac{5}{4}e^{-5x} + \frac{11}{4}e^{3x} \).

(iii) \[ \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0 \] when \( y = 3 \) and \( \frac{dy}{dx} = 7 \) when \( x = 0 \).

The auxiliary equation is \( \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0 \) when \( \lambda = 0, 2 \).

The solution to the differential equation is thus \( y = Ae^{0x} + Be^{2x} = A + Be^{2x} \) for constants A, B.

When \( x = 0 \), \( y = 3 \) so \( 3 = A + B \).

Now \( \frac{dy}{dx} = 2Be^{2x} \).

When \( x = 0 \), \( \frac{dy}{dx} = 7 \) so \( 2B = 7 \). Hence \( B = \frac{7}{2} \).

Substituting into \( A + B = 3 \) gives \( A = \frac{1}{2} \).

Hence the solution is \( y = \frac{7}{2} - \frac{1}{2}e^{2x} \).
Exercise 8:
\[
\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0
\]
so the auxiliary equation is \(\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0\) when \(\lambda = 1, 2\).

The solution to the differential equation is thus \(y = Ae^x + Be^{2x}\) for constants \(A, B\).

When \(x = 0\), \(y = 1\) so \(1 = A + B\).
When \(x = 1\), \(y = 2\) so \(2 = Ae + Be^2\).

So we must solve the following system of equations:
\[
\begin{align*}
A + B &= 1 \\
Ae + Be^2 &= 2
\end{align*}
\]

Multiplying the first equation by \(e\) and subtracting the second we get \(Be - Be^2 = e - 2\).

\[
\therefore (e - e^2)B = e - 2 \text{ and so } B = -\frac{e - 2}{e^2 - e}.
\]

Substituting into the first equation gives \(A = 1 + \frac{e - 2}{e^2 - e} = \frac{e^2 - 2}{e^2 - e}\).

Hence the solution is \(y = \left(\frac{e^2 - 2}{e^2 - e}\right)e^x - \left(\frac{e - 2}{e^2 - e}\right)e^{2x}\).

When \(x = -1\), \(y = \left(\frac{e^2 - 2}{e^3 - e}\right)e^{-1} - \left(\frac{e - 2}{e^3 - e}\right)e^{-2} = \frac{e - 2e^{-1} - e^{-1} + 2e^{-2}}{e^2 - e} = \frac{e - 3e^{-1} + 2e^{-2}}{e(e - 1)}\)

\[
= \frac{e^3 - 3e + 2}{e^3(e - 1)} = \frac{(e - 1)(e^2 + e - 2)}{e^3(e - 1)} = \frac{e^2 + e - 2}{e^3} \approx 0.40364.
\]