10. LOG AND EXPONENTIAL FUNCTIONS

10.1 What Do We Mean by $2^x$?

We know what $2^n$ means if $n$ is a positive integer. Just multiply $n$ factors of 2 together.

Negative integer powers are defined by taking reciprocals: $2^{-n} = \frac{1}{2^n}$.

Fractional powers are defined by taking $n$'th roots: $2^{m/n}$ means the $n$'th root of $2^m$, that is, the number which, when raised to the $n$'th power, gives $2^m$.

Example 1: $2^{-3} = \frac{1}{8}$, $2^{3/2} = \sqrt{8}$, and $2^{-3/2} = \frac{1}{\sqrt{8}}$.

But what does $2^x$ mean for an irrational numbers? What, for example, does $2^{\sqrt{2}}$ mean? The answer lies in the concept of limits. Although $\sqrt{2}$ is irrational it can be approximated by rational numbers. If we write out the decimal expansion of $\sqrt{2}$, 1.41421356…., we can take the sequence 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, 1.414213, …

Each of these is a rational number. We could write the sequence as:

1, $\frac{14}{10}$, $\frac{141}{100}$, $\frac{1414}{1000}$, $\frac{14142}{10000}$, $\frac{141421}{100000}$, ….

Because all of these are rational numbers we can define:

$2^1$, $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, $2^{1.4142}$, $2^{1.41421}$, $2^{1.414213}$, …

in terms of $10^n$ roots of $2^{14}$, $100^n$ roots of $2^{141}$ and so on.

It can be shown that this sequence of real numbers will approach a limit, and that limit is defined to be the value of $2^{\sqrt{2}}$. There are many other sequences of rational numbers that approach $\sqrt{2}$ and it can be shown that if we raise 2 to each of them we’ll get a sequence that approaches the same limit as before.

We define $2^x$ as the limit of:

$2^{x_1}$, $2^{x_2}$, $2^{x_3}$, ….

for any sequence $x_1$, $x_2$, $x_3$, … that approaches $x$.

This sounds all very technical. Anything to do with limits does get rather hard going. In practice if you had to draw the graph of $y = 2^x$ you’d plot the points for certain rational values of $x$ and join these points by a smooth curve.

In the same way we can define $a^x$ for all real numbers $a > 0$. (Of course $1^x$ equals 1 for all values of $x$.)

10.2 Differentiating $2^x$.

We can differentiate $y = x^2$ but what about $y = 2^x$? Here’s a rough picture of what the graph of $y = 2^x$ looks like.
As \( x \) becomes larger and larger \( 2^x \) grows exponentially. As \( x \) approaches \(-\infty\), \( 2^x \) approaches zero. For example \( 2^{-100} = \frac{1}{2^{100}} \) which is 1 over a very large number. It’s also important to realise that \( 2^x \) is always positive. (So don’t waste your time using Newton’s Method to solve the equation \( 2^x = 0 \). You can see geometrically what’s going to happen.)

What’s the slope at \( x = 0 \)? Our graph isn’t very accurate, but at least you can see that it must be positive, because \( y \) is increasing with \( x \).

When \( x = 0, y = 2^0 = 1 \). Let’s take a point very close by, say \( x = 0.001 \). Using a calculator we can discover that the corresponding value of \( y \) is, to 8 decimal places, 1.00069334. The slope of the chord joining these points is:

\[
\frac{1.00069334 - 1}{0.001 - 0} = \frac{0.00069334}{0.001} = 0.69334.
\]

Now this chord closely approximates the tangent so a good estimate for the slope of the curve at \( x = 0 \) is 0.69334.

But what if we want a general formula for the derivative of \( y = 2^x \) in the same way that we know that the derivative of \( y = x^2 \) is \( 2x \)? In this case, since none of the formulae we’ve developed so far apply, we’ll have to go back to first principles, using \( \Delta x \) and \( \Delta y \).

Let \( y = 2^x \) and let \( x \rightarrow x + \Delta x \) and \( y \rightarrow y + \Delta y \). Then \( y + \Delta y = 2^{x+\Delta x} = 2^x.2^{\Delta x} \). Since \( y = 2^x \) we have \( \Delta y = 2^x(2^{\Delta x} - 1) \).

\[
\frac{\Delta y}{\Delta x} = 2^x \frac{(2^{\Delta x} - 1)}{\Delta x}.
\]

Notice that we’ve written this as a product of two quantities where the second factor, \( 2^x \), only involves \( x \) and the first factor only involves \( \Delta x \). Now you may say that you can see some \( x \)’s inside that first bracket, but you’re just imagining it. The increment \( \Delta x \) is a single quantity. It isn’t a number \( \Delta \) multiplied by \( x \). We could have rewritten \( \Delta x \) as \( h \) and then it would be quite obvious that the only place where \( x \) appears is in the \( 2^x \) bit.

Indeed, let’s write \( \Delta x \) as \( h \). Then:

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{2^x(2^h - 1)}{h} = 2^x \lim_{h \to 0} \left( \frac{2^h - 1}{h} \right).
\]

Now what is \( \lim_{h \to 0} \left( \frac{2^h - 1}{h} \right) \)? For a start since it doesn’t have any \( x \)’s in it, it’s a constant, a particular number.

This constant is, in fact, the limit of the slope of the chord joining (0, 1) to \((h, 2^h)\) as \( h \) approaches zero, that is, as the second point approaches the first. And that’s just the slope of the tangent to \( y = 2^x \) at \( x = 0 \), roughly 0.693315. To get a good approximation to this, take a very small value of \( h \), say \( h = 0.00001 \), and calculate \( \frac{2^{0.00001} - 1}{0.00001} \).

So the derivative of \( 2^x \), with respect to \( x \), is \( k2^x \) where \( k \) is a constant, roughly equal to 0.693315.

10.3 The Number \( e \) and the Function \( e^x \).

We’ve differentiated \( y = 2^x \). What about \( y = a^x \) in general? A similar argument to the above shows that the derivative of \( a^x \) is a constant times \( a^x \). But don’t expect it to be the same constant. The constant will in fact be the slope of the chord joining \((0, 1)\) to \((h, 2^h)\) as \( h \) approaches zero, that is, as the second point approaches the first. And that’s just the slope of the tangent to \( y = a^x \) at \( x = 0 \), as “a” increases the graph of \( y = a^x \) climbs more steeply, so we expect the constant to increase with “a”.

The constant for \( a = 2 \) is about 0.693315. I wonder what it is for \( a = 3 \)? We can get a good approximation of it by calculating the slope of the chord joining the points \((0, 1)\) and \((0.001, 3^{0.001})\). This is:

\[
\frac{3^{0.001} - 1}{0.001} \approx \frac{1.001099216 - 1}{0.001} \approx \frac{0.001099216}{0.001} \approx 1.09922.
\]
So \( \frac{d(a^x)}{dx} = ka^x \) where \( k \) depends on “\( a \)” but not on “\( x \)”. The approximate values of \( k \) for \( a = 2 \) and \( a = 3 \) are as follows:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.69334</td>
</tr>
<tr>
<td>3</td>
<td>1.09922</td>
</tr>
</tbody>
</table>

It seems reasonable to assume that as the value of \( a \) increases so does \( k \), and that it does so in a continuous fashion. So somewhere between \( a = 2 \) and \( a = 3 \) the value of \( k \) becomes exactly 1. For this value of “\( a \)” it will be the case that the derivative of \( a^x \) is \( a^x \). Here we have a function that is its own derivative. At every point on the graph of this function, the slope will be exactly the same as the height.

So what is this special number? It’s important enough to be given its own special symbol. It’s called “\( e \)”. Now “\( e \)” is just a letter of the alphabet, like \( x \), \( y \) or \( z \), which can represent a variable. Not any more. Because “\( e \)”, the constant, is so important in mathematics we avoid using it for a variable.

There’s another constant in mathematics that’s just as important. It’s the number \( \pi \), the ratio of the circumference of a circle to the diameter and it enters into many formulae, such as \( \pi r^2 \) for the area of a circle with radius “\( r \)”.

The numbers \( \pi \) and “\( e \)” are the two most important constants in all of mathematics. The fact that one is denoted by a Greek letter and the other by an ordinary one is of no significance. They are numbers which keep popping up all over the place.

Like \( \pi \), the number “\( e \)” is irrational. That is it can’t be written as a ratio of two whole numbers. Never mind that some people say that \( \pi = \frac{22}{7} \). That’s just a convenient approximation.

Every number can be approximated by a fraction and if we wanted to we could say that \( e = \frac{19}{7} \).

We’d be wrong, but it is a moderately good approximation.

Well how big is “\( e \)”? Clearly it’s between 2 and 3.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>1.00000</td>
</tr>
<tr>
<td>3</td>
<td>1.09922</td>
</tr>
</tbody>
</table>

Also, since 1.09922 is closer to 1 than 0.69334, it’s likely that “\( e \)” is closer to 3 than it is to 2. In fact, to 8 decimal places, \( e \approx 2.71828183 \).

This function \( f(x) = e^x \) is called the exponential function and it’s one of the most important functions in mathematics. Since “\( e \)” was chosen so that the constant \( k \) is exactly 1 we can say that:

\[
\frac{d(e^x)}{dx} = e^x
\]

So what is \( \frac{d^2(e^x)}{dx^2} \), the second derivative of \( e^x \)? The answer, of course, is \( e^x \). If it doesn’t change when you differentiate it once, it won’t change when you differentiate it any number of times. So, for example, \( \frac{d^{10}(e^x)}{dx^{10}} = e^x \).

Integrating \( e^x \) is just as easy. What can you think of that gives \( e^x \) when you differentiate it? Yes, that’s right: \( e^x \). What doesn’t change in one direction doesn’t change in the other. So:

\[
\int e^x \, dx = e^x + c
\]
10.4 The Logarithmic Function

Every positive number is a power of 2. For example $2^{3.321928095}$ is very, very close to 10 and there’s some number $y$, very, very close to 3.321928095 where $2^y$ is exactly equal to 10. This number is called “the logarithm of x to the base 2”.

The number $y$ for which $2^y = x$ is called the logarithm of $x$ to the base 2. It’s written $y = \log_2 x$.

Example 2: Find $\log_2 128$.
Solution: Since $2^7 = 128$ it follows that $\log_2 128 = 7$.

Example 3: Find $\log_2 (1/8)$.
Solution: Since $\frac{1}{8} = 2^{-3}$ it follows that $\log_2 (1/8) = -3$.

Definition: If $b > 1$, the logarithm of $x$ to the base $b$ is that power of $b$ which exactly equals $x$.
Notation: It is written $\log_b x$.

A useful slogan to remember is:

**Logs are powers and powers are logs**

Example 4: Find $\log_{10} 1,000,000$.
Solution: One million is what power of 10? The answer is $10^6$. Logs are powers and powers are logs. So $\log_{10} 1,000,000 = 6$.

Example 5: Find $\log_3 1$.
Solution: $1 = 3^0$ so $\log_3 1 = 0$.

In fact, since $b^0 = 1$ for any base, the logarithm of 1 is zero, for any base.

Example 6: Find $\log_3 0$.
Solution: For what $y$ is $3^y = 0$? The answer is that there is no such $y$ and so $\log_3 0$ does not exist.

In fact, for any base $b$, $\log_b x$ is undefined if $x \leq 0$. That’s because $b^y$ is always positive. It can never equal zero or a negative number.

The most useful base in mathematics is the base “e”, logs to the base “e” are the most useful logarithms. They are called the natural logarithms.

Example 7: Find $\log_e(\sqrt{e})$.
Solution: Since $\sqrt{e} = e^{1/2}$ it follows that $\log_e(\sqrt{e}) = \frac{1}{2}$.

Usually, when doing calculus, we drop off the “e” altogether and write $\log x$, when we mean $\log_e x$. If we don’t tell you the base it’s assumed we mean base $e$. In some textbooks the author uses the notation LN(x) (short for “natural logarithm”) for the log of $x$ to the base $e$.

Example 8: Find $\log \frac{1}{\sqrt{e^3}}$.
Solution: “log” here means “log to the base e”. Since $\frac{1}{\sqrt{e^3}} = e^{-3/2}$ it follows that $\log \frac{1}{\sqrt{e^3}} = -\frac{3}{2}$.
10.5 Logs and Exponentials on Calculators

In our examples we’ve had to stick to examples where we could do the arithmetic in our heads. But generally we need a calculator.

$x^y$: For powers there’s an $x^y$ button. Use this, as follows, to find general powers.
(1) Input the base number $x$.
(2) Press the $x^y$ button.
(3) Now input the power $y$.
(4) Press the $=$ button.
The answer is now displayed.

**Example 9:** Find $3.56^{7.29}$.
**Solution:** $10472.98509$.

$e^x$: For powers of “$e$” you could use the $x^y$ button but you’d have to input the base “$e$” all the time. To save you the trouble most calculators have an $e^x$ button.

(1) Input $x$.
(2) Press the $e^x$ button.

On many calculators the $e^x$ is not written on a button but is written *above* the “LN” button. To get $e^x$ in such cases you have to press the [INV] key and follow it by the [LN] key.

If you can’t find an $e^x$ key you should have a key marked **exp**. Use this instead.
The answer is now displayed.

**Example 10:** Find $e^{2.9053}$.
**Solution:** $e^{2.9053} = 18.27072405$.

$log x$: By this we mean the natural logarithm, to the base “$e$”.
(1) Input $x$.
(2) Press the [LN] key.

If you don’t have one of these look for a key marked **log** and use this.

**Under no circumstances should you use the key marked the **log**. If you have such a key it refers to base 10 not base “$e$”.
(3) The answer is now displayed.

If you ever want logs to other bases you can get them by using the formula: $log_b x = \frac{log x}{log b}$. In the special case of logs to the base 10 you can just press the [log] key. If you don’t have one of these you’ll probably have a [log10] key. Use this instead.

**Example 11:** Find $log 16.8394$.
**Solution:** $log 16.8394 = 2.82372139$.

If you got 1.226326613 you must have pressed the “log” key instead of the log$_e$ or LN key.

10.6 Differentiating $log x$

Having introduced a new function we, of course, want to differentiate it. The technique we use here applies to any inverse function.
If \( g(x) \) is the inverse function to \( f(x) \) then \( f(g(x)) = x \) and \( g(f(x)) = x \). Doing one function followed by its inverse brings you back to the original “x”. That’s the case with the exponential function \( e^x \) and the log function \( \log x \).

What is \( \log(e^x) \)? Remember “logs are powers”. What power of “e” is equal to \( e^x \)? That’s easy, \( e^x = e^x \). So the power, and hence the log, is \( x \). That is:

\[
\log(e^x) = x
\]

And what’s \( e^{\log x} \)? Well, \( \log x \) is that power such that “e” raised to it gives “x”. So if you raise “e” to \( \log x \) you’ll get \( x \). That is:

\[
e^{\log x} = x
\]

The first of these equations holds for all \( x \). There’s no problem with \( \log(e^x) \) because \( e^x > 0 \) for all \( x \). But the second of these equations only holds for all positive \( x \). It doesn’t make sense if \( x \) is zero or negative. Because of difficulties like this we’ll suppose, throughout the rest of this section, that \( x > 0 \).

Now if \( y = \log x \) we can write this as \( x = e^y \). And we can now differentiate “x” with respect to “y” to get \( \frac{dx}{dy} = e^y \). This is just the familiar \( \frac{dy}{dx} = e^x \) when \( y = e^x \). It’s just that the roles of “x” and “y” have swapped.

Now if \( \frac{dx}{dy} = e^y \) what is \( \frac{dy}{dx} \)? Do we simply turn \( \frac{dx}{dy} \) upside down like we do with any old fraction? Wait! \( \frac{dx}{dy} \) isn’t a fraction. It’s the limit of a fraction.

But \( \frac{\Delta x}{\Delta y} \) is a genuine fraction, and so is \( \frac{\Delta y}{\Delta x} \). As fractions, it is true that \( \frac{\Delta x}{\Delta y} \cdot \frac{\Delta y}{\Delta x} = 1 \). Taking limits as \( \Delta x \rightarrow 0 \), and using the fact that the limit of a product is the product of the limits (true but we won’t prove it here) we get \( \frac{dx}{dy} \cdot \frac{dy}{dx} = 1 \).

This is true no matter what the functions are.

If \( x \) and \( y \) are related by a function and the inverse of that function then

\[
\frac{dx}{dy} \cdot \frac{dy}{dx} = 1.
\]

Getting back to \( y = \log x \), we have \( x = e^y \). So \( \frac{dx}{dy} = e^y = x \).

It follows that \( \frac{dy}{dx} = 1 \) so \( \frac{dy}{dx} = \frac{1}{x} \).

\[
\frac{d(\log x)}{dx} = \frac{1}{x}
\]

Every new derivative is a new integral, because differentiation and integration are inverse operations. So the fact that the derivative of \( \log x \) is \( \frac{1}{x} \) means that we can now integrate \( \frac{1}{x} \).

\[
\int \frac{dx}{x} = \log x + c
\]

provided the integral is over some interval where \( x > 0 \).
Here we’ve found the missing link. We’re able to integrate \( x^n \) for every \( n \), by the Power Rule, except for \( n = -1 \). This result completes the story.

\[
\int x^n \, dx = \begin{cases} 
\frac{1}{n+1} x^{n+1} + c & \text{if } n \neq -1 \\
\log x + c & \text{if } n = -1
\end{cases}
\]

### 10.7 Power Series

A polynomial is a \textit{finite} sum of constant multiples of powers of \( x \), such as

\[1 + 3x - 4x^2 + x^4.\]

(Normally we start at the highest power and we’d usually write this as \( x^4 - 4x^2 + 3x + 1 \).

We can differentiate polynomial functions term by term.

Sometimes it can make sense to have infinitely many terms. A famous example is the sum:

\[1 + x + x^2 + x^3 + \ldots\]

This is the sum to infinity of a geometric progression with common ratio \( x \) and it converges when \(|x| < 1\). If \(|x| < 1\) we write the sum to infinity as \( \frac{1}{1-x} \).

So we can write \( \frac{1}{1-x} = 1 + x + x^2 + x^3 \ldots \) (provided \(|x| < 1\)).

Now the exponential function \( e^x \) can also be written as an infinite series.

Suppose \( e^x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \)

Putting \( x = 0 \) we conclude that \( a_0 = 1 \).

Now differentiate both sides. This raises the question of whether it’s permissible to differentiate an infinite sum term by term just like we do for a finite sum. But because we don’t want to get embroiled in technical difficulties we’ll suppose that we can. There are some restrictions on when this is possible but in this case it’s perfectly permissible. However you’ll have to take that on authority.

So differentiating term by term we get:

\[ e^x = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots \]

Comparing the two series we equate corresponding coefficients. Is this permitted? It is, but we won’t stop for permission.

So we get:

\[ 1 = a_0 = a_1 \]

\[ a_1 = 2a_2 \]

\[ a_2 = 3a_3 \]

\[ a_3 = 4a_4 \] and so on.

So \( a_0 = a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{a_2}{3} = \frac{1}{6}, a_4 = \frac{a_3}{4} = \frac{1}{24}, \ldots \)

There is a very definite pattern to these denominators. They are factorials. Recall that factorial \( n \), written \( n! \), is the product of all the integers from 1 up to \( n \).

\[ 2! = 1.2 = 2 \]

\[ 3! = 1.2.3 = 6 \]

\[ 4! = 1.2.3.4 = 24 \] and so on.

So we can write the exponential function as an infinite series:

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots \]
Now this series converges for all \( x \). It can be used to find many of the properties of the exponential function.

You may not realise it but you carry this infinite series around with you inside your calculator. Have you ever wondered how a calculator can work out \( e^x \) when you press the appropriate key?

Basically a scientific calculator can only perform the four basic arithmetic operations of addition, subtraction, multiplication and division. Everything else it does has to be broken down into these four basic operations.

Whenever you press the \( e^x \) key it feeds the number into the above infinite series. Well, of course it can’t work out all the terms, so it just stops after a certain number of terms. The exponential series converges so rapidly that the sum of all the remaining terms, after a certain point, is negligible.

**Example 12:** Use the power series for \( e^x \) to approximate \( e \).

**Solution:**

\[ e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} + \ldots \]

We can set up our working in a spreadsheet. We start with \( 0! = 1 \) (it’s hard to imagine a product of all the numbers from 1 up to 0 so just accept the standard convention that \( 0! = 1 \)). We then divide each number in the \( 1/n! \) column by the next number in the \( n \) column. So 0.5 divided by 3 is 0.16666667 and 0.16666667 divided by 4 is 0.0416667.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 1/n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.16666667</td>
</tr>
<tr>
<td>4</td>
<td>0.04166667</td>
</tr>
<tr>
<td>5</td>
<td>0.00833333</td>
</tr>
<tr>
<td>6</td>
<td>0.00138889</td>
</tr>
<tr>
<td>7</td>
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</tr>
<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>0.00000276</td>
</tr>
<tr>
<td>10</td>
<td>0.00000028</td>
</tr>
</tbody>
</table>

Suppose we stop there, and add the terms. We get \( e \approx 2.71828181 \). My calculator gives \( e \approx 2.718281828 \) so it’s probably using a few more terms.

**EXERCISES FOR CHAPTER 10**

**Exercise 1:** Find \( \frac{dy}{dx} \) for each of the following:

(i) \( y = 2\log x + 3 \);
(ii) \( y = \log(x^2 + 1) \);
(iii) \( y = x\log x \)

**Exercise 2:** Use Exercise 1(iii) to find \( \int \log x \, dx \).

**Exercise 3:** Find the equation of the tangent to \( y = \log x \) at \( x = 2 \).

**Exercise 4:** Find the equation of the tangent to \( y = \log x \) that passes through \( (1, 0) \).

**Exercise 5:** Find the stationary points of \( y = x^2\log x \) and determine their nature.
**Exercise 6:** If \( z = x \log y \) find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

**Exercise 7:** Find the stationary points of \( z = \frac{\log y}{x} + y \).

**Exercise 8:** Evaluate the following definite integrals:

\[
\begin{align*}
(i) \quad & \int_{1}^{3} \frac{e}{x} \, dx ; \\
(ii) \quad & \int_{0}^{1} \frac{e^{-1}}{x + 1} \, dx .
\end{align*}
\]

**Exercise 9:** (a) Use Simpson’s Rule to estimate the integral \( \int_{1}^{5} \frac{dx}{x} \) using 4 strips.

(b) Evaluate \( \int_{1}^{5} \frac{1}{x} \, dx \) exactly, by integration.

**Exercise 10:**

(a) Find the sum to infinity of the geometric series:

\[
1 - x + x^2 - x^3 + x^4 - x^5 + \ldots
\]

(b) Integrate the sum to infinity, with respect to \( x \).

(c) Integrate the geometric series with respect to \( x \), term by term.

(d) Use (b) and (c) to get an infinite series for \( \log(1 + x) \).

(e) Use (d) to get an infinite series for \( \log \frac{1}{2} \).

(f) Use the first six terms of this series to get an estimate for \( \log \frac{1}{2} \).

(g) Hence get an estimate for \( \log 2 \). Compare this with the value that you obtain from your calculator.

**Exercise 11:**

(a) Show that the stationary points of \( y = \frac{1}{4} x^2 - x \log x \) occur when \( \log x = \frac{1}{2} x - 1 \).

(b) Use Newton’s Method to solve this equation to six decimal places.

[Hint: Start at various positive integer values of \( x \) until you get one to converge. Now try a different positive integer until you get one to converge to a different value. This time try a small positive fraction, between 0 and 1. There are two solutions to this equation.]

(c) Use the Second Derivative Test to determine the nature of each stationary point.
Exercise 1:

(i) \( \frac{2}{x} \).

(ii) \( y = \log(x^2 + 1) \);

\[ \frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x \] (chain rule) = \( \frac{2x}{x^2 + 1} \).

(iii) \( y = x \cdot \log x \)

\[ \frac{dy}{dx} = \log x + x \cdot \frac{1}{x} \] (product rule) = \( 1 + \log x \).

Exercise 2:

Since the derivative of \( x \cdot \log x \) is \( 1 + \log x \), it follows from the Fundamental Theorem of Calculus (integration is the reverse process of differentiation) that \( \int (1 + \log x) \, dx = x \cdot \log x \).

Hence \( \int 1 \, dx + \int \log x = x \cdot \log x \).

But \( \int 1 \, dx = x \), so \( \int \log x = x \cdot \log x - x + c \).

Exercise 3:

\( \frac{dy}{dx} = \frac{1}{x} = \frac{1}{2} \) when \( x = 2 \). So the tangent has slope \( \frac{1}{2} \). It passes through \((2, \log 2)\) so its equation is \( y - \log 2 = \frac{1}{2} (x - 2) = \frac{1}{2} x - 1 \). This can be simplified to \( y = \frac{1}{2} x + \log 2 - 1 \).

Exercise 4:

Let the point of contact be \((t, \log t)\). Since \( \frac{dy}{dx} = \frac{1}{x} \) the slope of the tangent is \( \frac{1}{t} \) and so its equation is: \( y - \log t = \frac{1}{t} (x - t) = \frac{x}{t} - 1 \).

When \( x = 0 \), \( y = 0 \) so \( -\log t = -1 \). Hence \( \log t = 1 \) and so \( t = e \).

The tangent is therefore \( y - \log e = \frac{x}{e} - 1 \), which can be simplified to \( y = \frac{x}{e} \).

Exercise 5:

\( \frac{dy}{dx} = 2x \cdot \log x + x^2 \cdot \frac{1}{x} = 2x \cdot \log x + x = x(2 \log x + 1) \).

\[ \frac{dy}{dx} = 0 \] when \( x = 0 \) or \( \log x = -\frac{1}{2} \).

However when \( x = 0 \) there is no value of \( y \) and hence no point on the curve.

So the only stationary point is when \( \log x = -\frac{1}{2} \). This corresponds to \( x = e^{-1/2} = \frac{1}{\sqrt{e}} \).

When \( x = \frac{1}{\sqrt{e}} \) then \( \log x < -\frac{1}{2} \) and so \( \frac{dy}{dx} = -- = + \).

When \( x = \frac{1}{\sqrt{e}} \) then \( \log x > -\frac{1}{2} \) and so \( \frac{dy}{dx} = + = - \).

Hence there is a local maximum at \( x = \frac{1}{\sqrt{e}} \).

Exercise 6: \( \frac{\partial z}{\partial x} = \log y \) and \( \frac{\partial z}{\partial y} = \frac{x}{y} \).
Exercise 7: \( \frac{\partial z}{\partial x} = -\frac{\log y}{x^2} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{xy} + 1. \)

At a stationary point \( \frac{\partial z}{\partial x} = 0 \) and \( \frac{\partial z}{\partial y} = 0. \) Hence \( \log y = 0 \) and \( xy = -1. \)

If \( \log y = 0 \) then \( y = 1 \) and from the other equation we have \( x = -1. \)

There is one stationary point at \((-1, 1).\)

Exercise 8:

\( e \)

(i) \( \int_3^1 \frac{e}{x} \, dx = [3\log x]_1^e = 3\log e - 3\log 1 = 3 - 0 = 3. \)

(ii) \( \int_0^{1/2} \frac{1}{x + 1} \, dx = [\log (x+1)]_0^1 = \log e - \log 1 = 1 - 0 = 1. \)

Exercise 9:

(a)

\[
\begin{array}{cccc}
 x & y & w & wy \\
 1 & 1 & 1 & 1 \\
 2 & 0.5 & 4 & 2 \\
 3 & 0.33333 & 2 & 0.66667 \\
 4 & 0.25 & 4 & 1 \\
 5 & 0.2 & 1 & 0.2 \\
\end{array}
\]

\[
\text{TOTAL} \quad 4.86667 \\
\text{INTEGRAL} \quad 1.622222
\]

(b) \( \int_0^5 \frac{1}{x} \, dx = \log 5 - \log 1 = \log 5 \approx 1.609438. \)

Although Simpson’s Rule agrees more or less with the exact value it’s not very accurate. This is a case where we definitely need more strips.

Exercise 10:

(a) \( 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \) is a geometric series whose first term is \( a = 1 \) and whose common ratio is \( r = -x. \) The sum to infinity is \( \frac{a}{1-r} = \frac{1}{1+x}. \)

(b) \( \int \frac{1}{1+x} \, dx = \log(1+x). \)

(c) \( \int (1 - x + x^2 - x^3 + x^4 - x^5 + \ldots) \, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots \)

(d) Hence \( \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots \)

Now this equation doesn’t hold for all values of \( x. \) After all the sum to infinity of the geometric series is only valid if \( x \) lies between \(-1 \) and \(+1.\)

(e) Substitute \( x = -\frac{1}{2} \) in (d):
\[ \log \frac{1}{2} = \left( \frac{-1/2}{2} \right) + \left( \frac{-1/2}{3} \right) \cdot -\frac{1}{8} + \left( \frac{-1/2}{4} \right) \cdot -\frac{1}{24} + \left( \frac{-1/2}{5} \right) \cdot -\frac{1}{64} + \ldots \]

(f) \( \log\frac{1}{2} \approx -(0.5 + 0.125 + 0.015625 + 0.00625) \approx -0.688542. \)

(g) \( \log\frac{1}{2} = -\log 2 \) so \( \log 2 \approx 0.688542. \) From a calculator we get \( \log 2 \approx 0.693147 \ldots. \)

The estimate is not particularly good, is it? The problem is simply that the series for \( \log(1 + x) \) that we obtained in (d) converges fairly slowly. To get an estimate that agrees with our calculator we would need to use many thousands of terms. Your calculator uses an inbuilt series to work out \( \log 2, \) but it uses one that converges much more quickly.

Exercise 11:

(a) \( y = \frac{1}{4} x^2 - x \log x \)
\[ \Rightarrow \frac{dy}{dx} = \frac{1}{2} x - (\frac{1}{x} + \log x) = \frac{1}{2} x - 1 - \log x. \]
\[ \frac{dy}{dx} = 0 \text{ when } \log x = \frac{1}{2} x - 1. \]

(b) Let \( y = \log x - \frac{1}{2} x + 1. \) Then \( y' = \frac{1}{x} - \frac{1}{2}. \) Starting at \( x_0 = 0 \) is no good. Why? Nor is it any good starting at \( x_0 = 2. \) Why? After a little bit of experimenting we start at \( x_0 = \frac{1}{2}. \)

<table>
<thead>
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<th>( x )</th>
<th>( y )</th>
<th>( y' )</th>
<th>( q )</th>
</tr>
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<td>1.655535</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

This quickly converges to \( x \approx 0.463922. \)

Let’s now try starting at \( x_0 = 4. \)

<table>
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<th>( y )</th>
<th>( y' )</th>
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<tbody>
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<td>0.000000</td>
<td>-0.313318</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

So the two stationary points of the function are at approximately \( x = 0.463922 \) and \( 5.356694. \)

(c) \( \frac{dy}{dx} = \frac{1}{2} x - 1 - \log x. \)
\[ \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2} - \frac{1}{x}. \]

When \( x \approx 0.463922 < 2 \) then \( \frac{1}{x} > \frac{1}{2} \) so \( \frac{d^2y}{dx^2} < 0. \) Hence there’s a local maximum at approximately \( 0.463922. \)

When \( x \approx 5.35669422 > 2 \) then \( \frac{1}{x} < \frac{1}{2} \) so \( \frac{d^2y}{dx^2} > 0. \) Hence there’s a local minimum at approximately \( 5.35669422. \)