7. MATRICES AND SYSTEMS OF LINEAR EQUATIONS

§7.1. Definition of a Matrix

Often data is best arranged in a table of rows and columns. Such a table can be considered as a single mathematical object, called a matrix (plural “matrices”). An \( m \times n \) matrix is a rectangular array of numbers arranged in \( m \) rows and \( n \) columns.

**Example 1:** \[
\begin{pmatrix}
1 & 5 & -7 \\
0 & 33 & -21
\end{pmatrix}
\]
is a \( 2 \times 3 \) matrix.

The entries in the table are called the **components** of the matrix. A general \( 2 \times 3 \) matrix can be written as \[
\begin{pmatrix}
a & b & c \\
d & e & f
\end{pmatrix},
\]
where the variables \( a, b, c, d, e, f \) are the components. But with larger matrices we might run out of letters of the alphabet. For this reason we use a double suffix notation. We denote the component in the \( i \)'th row and \( j \)'th column by the symbol \( a_{ij} \). Think of this as a single symbol rather than some combination of variables. It’s a bit like a name consisting of the family name and first and second given names. Just as Michael James McPherson is just one person so \( a_{25} \) is just one variable.

Of course with really large matrices we would have to put a comma between the \( i \) and the \( j \). The component \( a_{123} \) would be ambiguous. Is it the entry in the \( 12 \)'th row and \( 3 \)'rd column or the one in the \( 1 \)'st row and \( 23 \)'rd column. Really big matrices are often used in practice but calculations involving them would be done by a computer. In this case we would write the components as \( A(I, J) \) in our computer programs. But you will be practising with smaller matrices where the \( a_{ij} \) notation is adequate.

So a typical \( m \times n \) matrix would look like this:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}.
\]

If \( m = n \) we call \( A \) a **square matrix**.

A useful shorthand is to write \( A = (a_{ij}) \) where \( a_{ij} \) is the i-j component.

The **transpose** of \( A \) can then be written as \( A^T = (a_{ji}) \), that is, \( a_{ji} \), the component that was in the j-i position, is now in the i-j position. In other words \( A^T = \begin{pmatrix}
a_{11} & a_{21} & \ldots & a_{m1} \\
a_{12} & a_{22} & \ldots & a_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \ldots & a_{mn}
\end{pmatrix} \). Rows and columns have been interchanged. And if \( A \) was \( m \times n \) then \( A^T \) will be \( n \times m \).

Clearly \( A^{TT} = A \). If \( A^T = A \) we say that \( A \) is a **symmetric matrix**. It is symmetric about the diagonal that goes from top left to bottom right corners. (The other diagonal has no significance for matrices, so when we say the diagonal you know which one we mean.)

A **diagonal matrix** is a square matrix where every component that lies off the diagonal is zero.
Example 2: If \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \) then \( A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \).

\[ B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = B^T \text{ is a symmetric matrix.} \]

\[ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \text{ is a diagonal matrix.} \]

A **scalar matrix** is a diagonal matrix where all the diagonal components are equal. The special cases of scalar matrices are:

\[
0 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.
\]

The matrix 0 is called the **zero matrix** and I is the **identity matrix**.

§7.2. Adding and Subtracting Matrices

You can only add two matrices if they are the same shape and size. In this case we just add corresponding components. If \( A = (a_{ij}) \) and \( B = (b_{ij}) \) are two \( m \times n \) matrices then \( A + B \) is the \( m \times n \) matrix \( (a_{ij} + b_{ij}) \).

**Example 3:**

\[
\begin{pmatrix} 3 & 1 & -1 \\ 0 & 7 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 3 \\ 6 & -3 & 7 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 2 \\ 6 & 4 & 5 \end{pmatrix}.
\]

Subtraction is just as easy. We simply subtract corresponding components. Note that if subtract a matrix from itself we get the zero matrix.

Ordinary numbers, such as the components of a matrix, are called **scalars**. Usually they are the real numbers we were familiar with at school. If you have learnt about complex numbers you can work with matrices “over the complex numbers”. This simply means that the components can be any complex number. There are many other number systems that can be used for the components.

The \( m \times n \) **zero matrix** is the matrix of this size where all the components are zero. They are denoted by the symbol 0, even though this notation does not reveal the dimensions. Clearly \( A + O = A \) for all \( m \times n \) matrices A, where O is the \( m \times n \) zero matrix.

It is very easy to multiply a matrix by a scalar. You just multiply every component by that scalar. So to find \( 2A \) you simply double every component of A. If \( A = (a_{ij}) \) then \( kA = (ka_{ij}) \). A special case is where we multiply a matrix by \(-1\). We write the result as \(-A\) instead of \((−1)A\).

**Example 4:**

\[
5 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \end{pmatrix} \quad \text{and} \quad -\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \end{pmatrix}.
\]
These operations are such simple extensions of the corresponding operations for scalars that the same rules of algebra apply. But beware. We are now about to multiply two matrices and the algebra we learnt at school for numbers breaks down for matrices.

§7.3. Matrix Multiplication

Example 5: Suppose a carpet supplier stocks 100 different types of tile at varying costs per square metre. The cost of laying the tiles might vary also, though not necessarily in proportion to the cost of the tiles. An inexpensive tile might be very small and be difficult to lay and might cost more per square metre to lay than an expensive one.

All the information about costs would be contained in databases that would be, in effect, a matrix. To simplify matters let’s suppose that there are only three types of tile, 1, 2 and 3. The pricing information could be expressed as a $2 \times 3$ matrix $(a_{it})$ where $a_{1t}$ is the cost per square metre of tile $t$, just for the tiles themselves, and $a_{2t}$ is the cost per square metre for laying tile $t$.

Suppose this cost matrix is $C = \begin{pmatrix} 80 & 100 & 120 \\ 30 & 20 & 40 \end{pmatrix}$. This means, for example that tile 2 costs $100$ per square metre plus $20$ for laying.

Let’s suppose, for the sake of simplicity, that there are 4 customers, each requiring more than one type of tile. Their requirements could be expressed by a $3 \times 4$ matrix $D = (d_{tj})$ where $d_{ij}$ is the number of square metres of tile $t$ that customer $j$ requires. We might call this the “demand matrix”.

Let’s suppose that the demand matrix is $D = \begin{pmatrix} 20 & 40 & 60 & 0 \\ 50 & 30 & 70 & 100 \\ 60 & 0 & 90 & 80 \end{pmatrix}$. This means, for example, that customer 3 wants 60m$^2$ of tile 1, 70m$^2$ of tile 2 and 70m$^2$ of tile 3. Customer 4 does not want any of tile 1. He wants 100m$^2$ of tile 2 and 80m$^2$ of tile 3.

As well as the information about the individual types of tiles, the invoice for each customer needs to show the total cost of tiles and the total cost of laying them. What are these costs for customer 3?

The tile cost will be $80 \times 60 = 4800$ for tile 1, $100 \times 70 = 7000$ for tile 2 and $120 \times 90 = 10800$ for tile 3, a total cost of $22600$.

The laying cost will be $30 \times 60 = 1800$ for tile 1, $20 \times 70 = 1400$ for tile 2 and $40 \times 90 = 3600$ for tile 3, a total cost of $6800$.

The total tile cost (in dollars) is $80.60 + 100.70 + 120.90 = 22600$.

The total laying cost (in dollars) is $30.60 + 20.70 + 40.90 = 6800$.

Similar calculations apply to the other customers.

Notice the pattern that lies behind these calculations. To get the tile cost for a particular customer we go along the first row of the cost matrix and simultaneously down the appropriate column of the demand matrix, multiplying corresponding numbers and adding the products. For the laying costs we do the same except that we go along the second row of the cost matrix.

We get two totals for each customer, the tile cost and the laying cost. These can be set out in a $2 \times 4$ matrix $T = (t_{ij})$ where $t_{ij}$ is the tile cost for customer $j$ and $t_{2j}$ is the laying cost.
This is how matrix multiplication works.

Matrix multiplication can be defined as follows. If A is an $m \times n$ matrix and B is an $n \times r$ matrix then we define $AB$ to be the matrix C where the i-j component of C is obtained by simultaneously running along the i'th row of A and down the j'th column of B, multiplying corresponding components and adding all these products.

You can see why the number of columns in A has to be the same as the number of rows in B. If they weren’t the same you would run out of a’s before you had finished with the b’s.

Example 6: Let us try finding $DC$ in example 5.

\[
DC = \begin{pmatrix} 20 & 40 & 60 & 0 \\ 50 & 30 & 70 & 100 \\ 60 & 0 & 90 & 80 \end{pmatrix} \begin{pmatrix} 80 & 100 & 120 \\ 30 & 20 & 40 \end{pmatrix} = \begin{pmatrix} 13800 & 6200 & 22600 & 19600 \\ 4000 & 1800 & 6800 & 5200 \end{pmatrix}.
\]

To get the 1-1 entry in the product we would have to run along the 1'st row of D and down the 1'st column of C, multiplying and adding: $20*80 + 40*30 + 60*? + 0*?$ You see the problem? This product is not defined.

Example 7: If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ find $AB$ and $BA$.

Solution: If two matrices are both $n \times n$ matrices we can multiply them in both orders, but we don’t usually get the same answer.

\[
AB = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} a + 2d + 3g & b + 2e + 3h & c + 2f + 3k \\ 4a + 5d + 6g & 4b + 5e + 6h & 4c + 5f + 6k \\ 7a + 8d + 9g & 7b + 8e + 9h & 7c + 8f + 9k \end{pmatrix}
\]

while

\[
BA = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} a + 4b + 7c & 2a + 5b + 8c & 3a + 6b + 9c \\ d + 4e + 7f & 2d + 5e + 8f & 3d + 6e + 9f \\ g + 4h + 7k & 2g + 5h + 8k & 3g + 6h + 9k \end{pmatrix}.
\]

We can express the rule for matrix multiplication using \(\Sigma\) notation. If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times r$ matrix then we define $AB$ to be the matrix $C = (c_{ij})$ where the components of $C$ are given by

\[
c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.
\]

The algebra of matrices is similar in some ways to the algebra of real numbers that you learnt in high school. But in some ways it is quite different. You will need to learn algebra all over again!

The $n \times n$ identity matrix is $I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$, with 1’s down the diagonal and zeros everywhere else. It is easy to see that this behave like the real number 1 in that $AI = IA = A$ for all $n \times n$ matrices $A$. 

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The following theorem can we proved fairly readily using Σ notation. However the proofs are somewhat tedious, and this is only an elementary course so we will be content with stating them.

**Theorem 1:** If A, B, C are matrices whose dimensions allow all the following sums and products to be computed then:

1. **Associative Law:** \((AB)C = A(BC)\).
2. **Distributive Laws:** \(A(B + C) = AB + AC\) and \((A + B)C = AC + BC\).

So far the algebra of matrices is behaving like the algebra of real numbers. But here are some important differences, even for \(2 \times 2\) matrices.

1. \(AB \neq BA\) in general. (There will be certain cases where \(AB = BA\) but these are exceptions.)
2. \(AB = 0\) does not necessarily mean that \(A = 0\) or \(B = 0\).

**Example 8:** Let \(A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}\) and \(B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\).

Then \(AB = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}\) even though neither \(A\) nor \(B\) is zero.

And \(BA = \begin{pmatrix} 0 & 0 \\ 7 & 0 \end{pmatrix}\) so \(AB \neq BA\).

Let’s think about some of the implications of the algebra of matrices failing in these respects.

1. We learnt in high school that \((A + B)^2 = A^2 + B^2 + 2AB\). Strictly speaking it should be \((A + B)^2 = A^2 + B^2 + AB + BA\) but in the algebra of real numbers we can sweep up \(AB\) and \(BA\) together as \(2AB\) because \(AB = BA\) for real numbers. But in the algebra of matrices the best we can do is \((A + B)^2 = A^2 + B^2 + AB + BA\).

2. The difference of two squares: \(A^2 - B^2 = (A - B)(A + B)\). This is OK for real numbers but not so for matrices. If you expand \((A - B)(A + B)\) you get \(A^2 - BA + AB - B^2\). If \(AB \neq BA\), which is the usual case for matrices, then we can’t cancel to get \(A^2 - B^2\).

3. In high school algebra we learn that quadratic equations have two solutions. Well, sometimes the two solutions coincide and there is only one solution. And if we do not want to stray into the field of complex numbers there are certain real quadratics that have no solution. But never more than two. After all a polynomial of degree 2 can have at most n zeros.

   In particular a real number has at most two square roots. There are two square roots of 1, namely \(\pm 1\), and we can prove it using algebra.

   Let \(x^2 = 1\).
   \[\therefore x^2 - 1 = 0.\]
   \[\therefore (x - 1)(x + 1) = 0\]
   \[\therefore x = \pm 1.\]
The $2 \times 2$ matrix that corresponds to the real number $0$ is the zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and corresponding to the real number $1$ we have the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let us translate the above proof into the notation for matrices and consider how much of it is valid for matrices.

Let $X^2 = I$.

$\therefore X^2 - I = O$.

$\therefore (X - I)(X + I) = O$.

$\therefore X = \pm I$.

The first step is just an assumption so it is OK. We then subtract $I$ from both sides. This is perfectly permissible for matrices. In the third step we are factorising the difference of two squares. Alarm bells should start to ring because this doesn’t work in general for matrices. But it is a false alarm. Since $IX = XI$ we do get cancellation. All is correct down to the third line. It is in the last line that things go wrong. Just because a product of matrices is zero does not mean that one or other factor is zero. In fact there are infinitely many square roots of the $2 \times 2$ identity matrix.

**Example 9:** Square the matrix $X = \begin{pmatrix} a & 1 \\ 1 - a & -a \end{pmatrix}$.

**Solution:** $X^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(4) Closely related to (3) is the fact that we can always divide by any real number, except zero. That is because every non-zero real number has an inverse under multiplication, $1/x$. With matrices there are plenty of matrices that fail to have such an inverse.

**Example 10:** Show that the matrix $A = \begin{pmatrix} 6 & 8 \\ 9 & 12 \end{pmatrix}$ has no inverse under multiplication.

**Solution:** The simplest way to show this is to observe that if $v = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ then $Av = 0$.

If $A^{-1}$ existed be could multiply both sides by $A^{-1}$ to get $v = 0$. Since this is not true then it must be that $A^{-1}$ does not exist.

A square matrix $A$ is **invertible** (or “non-singular”) if it has an inverse under multiplication. The inverse $B$ has to be 2-sided, meaning that not only is $AB = I$ but $BA = I$. Some matrices which are not square have left inverses but no right inverse, or vice versa. But if a matrix has both a left and a right inverse they are equal.

For $2 \times 2$ matrices the test for whether $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible is that $ad - bc \neq 0$. If this is the case then $A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

**Example 11:** Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

**Solution:** The inverse is $-\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.
§7.4. Two Equations in Two Variables

Example 12: Find numbers x, y such that
\[
\begin{align*}
x + 2y &= 20 \\
5x - 2y &= 40
\end{align*}
\]

Solution: If we add the two equations we get
\[6x = 60,\] that is, \(x = 10.\)
What we have done is to combine the two equations in such a way that we have eliminated \(y.\)
We can then solve for \(x.\)
We now substitute \(x = 10\) into either of the original two equations to get \(y = 5.\)
So \(x = 10,\ y = 5\) is the required solution.

Example 13: Find numbers x, y such that
\[
\begin{align*}
7x + 5y &= -8 & \text{......... (1)} \\
5x - 2y &= 11 & \text{......... (2)}
\end{align*}
\]

Solution: Suppose we attempt to eliminate \(y.\) Take 2 times equation (1) and add 5 times equation (2). This will give \(14x + 25x = -16 + 55,\) that is, \(39x = 39.\)
Hence \(x = 1.\)
Substituting into equation (1) we get \(7 + 5y = -8,\) that is, \(5y = -15\) or \(y = -3.\)
So \(x = 1,\ y = -3\) is the required solution.

We could have substituted \(x = 1\) into equation (2) instead. In fact we could have eliminated \(x\) to get \(y = -3\) and then substitute to get \(x = 1.\)

§7.5. Geometric Interpretation

We can view what is going on by noting that a linear equation \(ax + by = c\) represents a straight line in the \(x-y\) plane. Two equations represents two straight lines and solving them simultaneously means finding the points that they have in common. The normal thing to happen is that two lines intersect in a single point.

However there are two other possibilities. The two lines might be parallel (but different), in which case there will be no common point and so no solution.

Or the two lines might coincide.

In this case the whole line will be in common and so there will be infinitely many solutions.
Example 14: Solve the system
\[ 6x + 9y = 10 \] .................................. (1)
\[ 8x + 12y = 7 \] .................................. (2)
**Solution:** Multiply (1) by 8.
\[ 48x + 72y = 80 \] .................................. (3)
Multiply (2) by 6.
\[ 48x + 72y = 42 \] .................................. (5)
Comparing (3) and (5) we get a contradiction. Hence there is no solution.

Example 15: Solve the system
\[ 6x + 9y = 3 \] .................................. (1)
\[ 8x + 12y = 4 \] .................................. (2)
**Solution:** Multiply (1) by 8.
\[ 48x + 72y = 24 \] .................................. (3)
Multiply (2) by 6.
\[ 48x + 72y = 24 \] .................................. (5)
Comparing (3) and (5) we see that the system has reduced to one equation. This corresponds to the situation where the two lines coincide. Hence there are infinitely many solutions.

These two degenerate cases will be pretty obvious when we have two equations in two variables but when it comes to three equations in x, y, z degenerate cases may not be so obvious.

§7.6. Three Equations in Three Variables
The equation \( ax + by + cz = d \) represents a plane in 3-dimensional space. Three such equations represent three planes. The normal situation is for there to be just one point in common, in which the corresponding system has a unique solution.

But there is the possibility of no solution, even if no two planes are parallel.

Here the lines of intersection of the pairs of planes are all parallel. A system corresponding to this situation will have no solution. More obvious cases where there is no solution is where two planes (or even all three) are parallel but different.
Then there is the case where the planes are like the pages of a book – all passing through a single line.

A system of equations corresponding to this situation will have infinitely many solutions. Finally there are the more obvious cases where two planes, or all three planes coincide.

The technique to apply here is again elimination of variables. But you must be disciplined about it. If you are not careful you will find yourself eliminating one variable only to have it popping up again and start going around in circles. Decide which variable you are going to eliminate and stick to it. Use the first two equations to eliminate this variable. Then use the first and third equations to eliminate the same variable. You will now have two equations in two variables which you proceed to solve as above.

**Example 16:** Solve the system
\[
\begin{align*}
2x - 5y + 6z &= 10 \tag{1} \\
3x + 2y - 5z &= -8 \tag{2} \\
5x + 3y - 7z &= -10 \tag{3}
\end{align*}
\]

**Solution:** Let us eliminate \(x\).

\((1) \times 3 \text{ minus } (2) \times 2 \text{ gives} \)
\[-19y + 28z = 46 \tag{4}\]

\((1) \times 5 \text{ minus } (3) \times 2 \text{ gives} \)
\[-31y + 44z = 70 \tag{5}\]

\((4) \times 31 \text{ minus } (5) \times 19 \text{ gives} \)
\[32z = 96 \tag{6}\]

Hence \(z = 3\).

Substituting in (4) gives \(19y = 28 \times 3 - 46 = 38\), so \(y = 2\).

Substituting in (1) gives \(2x = 5 \times 2 - 6 \times 3 + 10 = 2\), so \(x = 1\).

§7.7. Systems of Linear Equations and Matrices

It is possible to write a system of three equations in \(x, y, z\) as a single matrix-vector equation.

**Example 8:** Write the following system as a matrix-vector equation.
\[
\begin{align*}
2x - 5y + 6z &= 10 \tag{1} \\
3x + 2y - 5z &= -8 \tag{2} \\
5x + 3y - 7z &= -10 \tag{3}
\end{align*}
\]

**Solution:** Let \(\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\) and let \(\mathbf{c} = \begin{pmatrix} 10 \\ -8 \\ -10 \end{pmatrix}\).

Let \(A\) be the \(3 \times 3\) matrix of coefficients \[
\begin{pmatrix} 2 & -5 & 6 \\ 3 & 2 & -5 \\ 5 & 3 & -7 \end{pmatrix}\]
Then \( A v = \begin{pmatrix} 2 & -5 & 6 \\ 3 & 2 & -5 \\ 5 & 3 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 5y + 6z \\ 3x + 2y - 5z \\ 5x + 3y - 7z \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ -10 \end{pmatrix} = c. \)

Now if \( A \) had an inverse under multiplication we would get the unique solution \( v = A^{-1}c. \)

In the system given in example 8 it turns out that \( A^{-1} \) does exist and \( v = A^{-1}c \) gives the solution. But for the systems in examples 6 and 7 \( A^{-1} \) does not exist.

It seems that if we want to take solving systems of linear equations further we should learn more about matrices. This is indeed the case, though the most efficient method for solving such systems is a process called the Gaussian algorithm that is based on the basic idea of eliminating variables. But here is not the place to learn such things.

**EXERCISES FOR CHAPTER 7**

**Exercise 1:** If \( A = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 1 & 8 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & 2 \\ 0 & 7 & -3 \\ 6 & 1 \end{pmatrix} \), find \( 2A - B. \)

**Exercise 2:** If \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & 1 \\ 1 & 2 \\ 3 & 5 \end{pmatrix} \), find \( AB. \) Is \( AB = BA? \)

**Exercise 3:** If \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), find \( A^2 + 3A + I. \)

**Exercise 4:** If \( A = \begin{pmatrix} 0 & 3 \\ 1 & 5 \end{pmatrix} \) show that \( A - 3A^{-1} = 5I. \)

**SOLUTIONS FOR CHAPTER 7**

**Exercise 1:** \( 2A = \begin{pmatrix} 2 & 10 & -4 \\ 12 & 2 & 16 \end{pmatrix} \) so \( 2A - B = \begin{pmatrix} -2 & 8 & -4 \\ 5 & 5 & 10 \end{pmatrix} \).

**Exercise 2:** \( AB = \begin{pmatrix} 4 & 2 & 9 \\ 16 & 5 & 18 \\ 4 & 10 & 30 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 39 & 44 \end{pmatrix}. \)
\( BA = \begin{pmatrix} 4 & 1 \\ 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 13 & 18 \\ 9 & 12 & 15 \\ 23 & 31 & 39 \end{pmatrix}. \)
So \( AB \neq BA \) because they are not even of the same size.

**Exercise 3:** \( A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \) so \( A^2 + 3A + I = \begin{pmatrix} 11 & 16 \\ 24 & 35 \end{pmatrix}. \)

**Exercise 4:** \( A^{-1} = -\frac{1}{3} \begin{pmatrix} 5 & -3 \\ -1 & 0 \end{pmatrix} \) so \( A - 3A^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 5I. \)