1. REAL NUMBERS

§1.1. Symbols Representing Numbers

In high school algebra we learnt that we can represent “unknown” numbers by letters such as x, y, z. These numbers are generally called variables, meaning that either their value is unknown, or it can vary. This process allowed us to express a numerical problem as an equation which we then solve.

Example 1: Find a number which when you double it and add 7 you get 15.
Solution: Let the number be x. Then 2x + 7 = 15.
Subtracting 7 from both sides of the equation we get 2x = 8.
Dividing both sides by 2 we get x = 4.

Generally we use single letters of the alphabet. We do not usually use pairs of letters because this could lead to confusion. Since we write b times c as bc we should not use “be” as a variable. But since there are only 26 letters of the alphabet we may need more than just a - z. To overcome this problem we use subscripts. If we have a list of numbers we can write them as: x₁, x₂, x₃, ...

Unlike x², where the composite symbol represents a combination of x and 2, variables with subscripts are to be treated as a single symbol. If you only know the value of x there is no way you can find the value of x₃. The symbols x and x₃ represent two, possibly unrelated, numbers. Think of x₃ as a name with “x” as the family name and “3” as the given name.

The subscripts need not start with “1”. Sometimes we find it convenient to have an x₀ in the family. It would even be possible to have x₋₁ as a variable but this would be more uncommon.

Although one subscript is sufficient for infinitely many variables when we have a lot of variables arranged in a table we usually use two suffixes. We could write the entry in the i’th row and j’th column as xᵢ,ⱼ. More usually we leave out the comma and write this as xᵢⱼ.

So x₂₅ would denote the number in the second row and fifth column. You have probably spotted a potential problem here. Is x₁₂₅ in the 12th row and 5th column or is it in the 1st row and 25th column? Clearly when there is danger of such confusion we must include the comma.

The objects that are represented by letters are numbers. These need not be whole numbers as in the above example. They can be fractions or, as we say, rational numbers. Or they can be any number on the number line, numbers we call real numbers. It is quite difficult to define a real number precisely. For now we can loosely define it to be something that represents a point on the number line.
An integer is a whole number, such as 17 or −8. A rational number is one that can be expressed in the form \(\frac{m}{n}\) where m, n are integers, such as \(\frac{3}{4}\) or \(-\frac{3}{7}\). One way of representing real numbers is by way of decimals.

**Example 2:**

\[
\begin{align*}
\frac{1}{4} &= 0.25 \\
-\frac{3}{5} &= -0.6 \\
137/10 &= 13.7
\end{align*}
\]

Many numbers require infinitely many decimal places to represent them exactly. (In view of more advanced mathematics we should say that “most” numbers have infinite decimal expansions.) In many of these cases the numbers have repeating decimal expansions. They can be written compactly by putting a dot over a repeating digit, or over the first and last digits of a repeating block of digits.

**Example 3:**

\[
\begin{align*}
2/3 &= 0.6666666666666666... = 0.6 \\
22/7 &= 3.14285714285714... = 3.142857
\end{align*}
\]

Numbers ending in \(\mathbf{9}\) can be represented in two ways.

**Example 4:** Show that \(3.569 = 3.57\).

**Solution:** Let \(x = 3.5699999...\)

Then \(10x = 35.69999...\)

Now \(x = 3.569999\)

Subtracting, \(9x = 32.13\).

Hence \(x = 3.57\).

It is often argued, wrongly, that 0.9999... is a tiny bit less than 1. But, using the above argument we can see that it is simply another way of writing 1. If you’re still not convinced ask yourself what number lies half way between 0.9999... and 1?

Many (most) real numbers have infinite decimal expansions that do not repeat. For example, the number \(\pi\) is the circumference of a circle whose diameter is 1. This number is so important in mathematics we use a special symbol for it. Unlike \(x\), \(\pi\) is not a variable. It is shorthand for a number that would take forever to write out exactly.

**Example 5:** \(\pi = 3.14159266358979323846264338327950288...\)

It can be shown that the decimal expansion of \(\pi\) is not repeating.

It is often claimed that \(\pi = 22/7\). This is not true, but 22/7 is a fairly good approximation as you can see from the expansion of 22/7 given above. A much better approximation is \(355/113 = 3.1415929...\)
Numbers that have finite decimal expansions, like $\frac{7}{16} = 0.4375$ are rational, but most rational numbers have infinite decimal expansions, such as $\frac{3}{7} = 0.285714285714285714 \ldots$

However it can be shown that rational numbers always have repeating decimal expansions while irrational numbers do not.

The number $\pi$ has been computed to over 5 trillion decimal places and no pattern has been detected. Certainly there is no sign of it repeating. But even 5 trillion decimal places cannot answer the question as to whether the expansion eventually repeats.

Mathematically it has been proved that $\pi$ is irrational. This means it can never be expressed exactly as a fraction, like $22/7$ or $355/133$. It also means that the decimal expansion of $\pi$ can never. This is a good example to show that computers, though they are an extremely useful tool in mathematics, will never replace mathematics. A computer, spewing out the digits of $\pi$, can never in finite time answer the question “do the digits of $\pi$ repeat?”

Yet a theorem in mathematics proves that the answer is “no”.

§1.2. The Laws of Algebra

There are certain properties that the real numbers possess which underlie basic algebra.

The Commutative Laws:
For all real numbers $a + b = b + a$ and $ab = ba$.

You have known for a long time that it does not matter in which order you add a list of numbers, and you can multiply numbers in any order. But be warned. This is not the usual state of affairs in mathematics, or in life for that matter. The order in which you carry out certain operations is often highly critical. If a watchmaker has to reassemble a watch it does matter in which order he puts in the pieces. Carrying out the operations of putting on your socks and your shoes you get quite a different result if you reverse the usual order!

There are mathematical objects which can be multiplied where $ab \neq ba$. This leads to all sorts of complications. So remember, despite the heading for this section we are not really discussing the laws of algebra in general – only the laws of the algebra of real numbers.

The Associative Laws:
For all real numbers $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

This means that when we add three or more terms (expressions or numbers that are added together), or multiply three or more factors (expressions or numbers that are multiplied together), it does not matter how we group them. As a result we usually write the above as $a + b + c$ and $abc$ respectively.

Again these are not laws that operate in all algebraic situations, though they do hold in most. When you come to learn about vector products you will discover a system that is not associative.

Because of the associative laws for real numbers we write things like $3x$, meaning $x + x + x$ and $x^3 = xxx$. If addition or multiplication were not associative these would be ambiguous. Does $3x$ mean $(x + x) + x$, or is it $x + (x + x)$? Fortunately for us it does not matter for real numbers.
Identities:
There are two special real numbers that behave specially when it comes to addition and multiplication. These are the numbers 0 and 1.
For all real numbers $x$, $0 + x = x + 0 = x$ and $1x = x1 = x$.
These numbers are called the **additive identity** (the number 0) and the **multiplicative identity** (the number 1).

Inverses:
For all real numbers $x$ there is a number, written $-x$ such that $x + (-x) = (-x) + x = 0$. The number $-x$ is called the **additive inverse** of $x$. When it comes to multiplication we have to make an exception. For all non-zero real numbers $x$ there is a number, written $1/x$ or $x^{-1}$, such that $x^{-1}x = x^{-1}x = 1$. It is called the multiplicative inverse.

Why the exception? Why can’t 0 have a multiplicative inverse? Why don’t we write $\frac{1}{0} = \infty$? Quite apart from the difficulty of finding a point on the real line to represent it, our whole system would implode if we allowed this. This is because we would then have $0\infty = 1$. But what’s wrong with that? The answer is “nothing, if you want to abandon the distributive laws”. We insist on having the distributive law, so 0 will just have to do without an inverse.
Let’s see what the distributive law is.

Distributive Law:
The distributive law ties the additive and multiplicative structures of the real numbers together.

For all real numbers $a(b + c) = ab + ac$.

Because multiplication is commutative we can also write $(b + c)a = ba + ca$. This is a fundamental property of the real numbers and we use it every time we expand an expression.

Example 7: Expand $(3x + 2y)(5x + y)$.
**Solution:**
$$(3x + 2y)(5x + y) = (3x + 2y)(5x) + (3x + 2y)(y)$$
$$= 15x^2 + 10xy + 3xy + 2y^2$$
$$= 15x^2 + 13xy + 2y^2.$$  

Here we are using more than the distributive law. When we multiplied $3x$ by $5x$ to get $15x^2$ we unconsciously made use of the associative law and the commutative law for multiplication. And the fact that we wrote the answer as $15x^2 + 13xy + 2y^2$ and not $(15x^2 + 13xy) + 2y^2$ or $15x^2 + (13xy + 2y^2)$ shows that we are mindful of the associative law for addition. Note too that in writing $10xy + 3xy$ as $13xy$ we are unconsciously making use of the distributive law. If we had to justify it we could write $10xy + 3xy = (10 + 3)xy = 13xy$.

When we worked with algebraic expressions in high school we probably didn’t think too much about what we were doing. We just instinctively changed one expression into an equivalent one. But if we want to know why things work the way they do we could justify everything from the above fundamental laws.

And what if we wanted a proof that these laws are correct? To do that we would have to carefully define a real number. Even defining the number 2 in a formal way is quite a sophisticated matter. Now is not the time or place to get bogged down with such fundamentals.

However we can provide a geometric explanation of the distributive law. If you accept that $ab$ is the area of an $a \times b$ rectangle then this picture demonstrates that

$$a(b + c) = ab + ac.$$
Someone interested in the foundations of mathematics would not be content with such a proof, but it will do for now.

Going back to why we can’t have \(0 \times \infty = 1\) if we have the distributive law, consider the following. Suppose \(0 \times \infty = 1\).

Then \(1 = 0 \times \infty\)
\[= (0 + 0) \times \infty\]
\[= 0 \times \infty + 0 \times \infty\]
\[= 1 + 1\]
\[= 2,\] which is nonsense.

Now there are many other laws of algebra that were not given in the above list. For example we all know that “if you multiply by zero you get zero”. Is this yet one more law we have to accept? No, it is a simple consequence of the ones we’ve already given.

**Example 8:** Show that \(0x = 0\) for all real numbers \(x\).

**Solution:** Let \(0x = y\).

Then \(y = 0x\)
\[= (0 + 0)x\]
\[= 0x + 0x\]
\[= y + y.\]

We would now cancel \(y\) on both sides to get \(y = 0\). But what is exactly going on when we cancel?

From \(y = y + y\) we get, by adding \(-y\) to both sides

\(0 = y + (-y) = y + y + (-y) = y + 0 = y.\)

Can you identify the basic laws used here?

Students are often perplexed as to why \(-1\) times \(-1\) is \(+1\). They are often fobbed off by teachers with some vague explanation such as “two negatives make a positive”. Certainly if something is not impossible then it is possible, but what has this to do with algebra? The more satisfactory explanation is the following.

**Example 9:** Show that \((-1)(-1) = 1\).

**Solution:** Let \(x = -1\).

Then \(1 + x = 0\) and so \(0 = (1 + x)x = x + x^2\).

So \(x^2 + x = 1 + x\) and hence \(x^2 = 1\).

It follows that the product of two negative numbers is positive, for example \((-3)(-2) = 6\). Perhaps the teacher who said “two negatives make a positive” meant this, but it is unlikely that he would have offered a proof. And in the early stages of algebra this would be appropriate.
Example 10: Show that if \( xy = 0 \) then either \( x = 0 \) or \( y = 0 \).
Solution: Suppose \( xy = 0 \).
If \( x \neq 0 \) then \( x^{-1} \) exists and we can multiply both sides by \( x^{-1} \) to get \( x^{-1}(xy) = 0 \).
By the associative law this becomes \( (x^{-1}x)y = 0 \). Hence \( 1y = y = 0 \).

Example 11: Show that if \( xy = xz \) and \( x \neq 0 \) then \( y = z \).
Solution: This is proved similarly.

§1.3. Basic Algebraic Identities
There are certain equations that hold for all variables in the algebra of real numbers. Here are a couple that are useful to know.

Theorem 1: For all real numbers \( x, y \):
\[
\begin{align*}
(1) \quad (x + y)^2 &= x^2 + 2xy + y^2; \\
(2) \quad (x - y)^2 &= x^2 - 2xy + y^2; \\
(3) \quad (x - y)(x + y) &= x^2 - y^2; \\
(4) \quad (x - y)(x^2 + xy + y^2) &= x^3 - y^3; \\
(5) \quad (x + y)(x^2 - xy + y^2) &= x^3 + y^3.
\end{align*}
\]

Proof:
(1) \( (x + y)^2 = (x + y)(x + y) \)
\[
= x(x + y) + y(x + y) \\
= x^2 + xy + yx + y^2 \\
= x^2 + xy + xy + y^2 \\
= x^2 + 2xy + y^2.
\]

(2) to (5) can be proved similarly.

The special cases of (3), (4) and (5), where \( y = 1 \), should be noted as they arise frequently.
\[
\begin{align*}
x^2 - 1 &= (x - 1)(x + 1); \\
x^3 - 1 &= (x - 1)(x^2 + x + 1); \\
x^3 + 1 &= (x + 1)(x^2 - x + 1).
\end{align*}
\]
Also note that although \( x^2 - 1 \) factorises, \( x^2 + 1 \) does not.

The first of these identities can be illustrated geometrically as follows.
Example 12: Factorise $x^4 - 1$.
Solution: $x^4 - 1 = (x^2 - 1)(x^2 + 1)$ by replacing $x$ by $x^2$ in the factorisation of $x^2 - 1 = (x - 1)(x + 1)(x^2 + 1)$.

§1.4. Solving Linear Equations
Solving an equation means to find all of the values of the variables that make the equation true. Some equations have no solutions, others have a single, or unique solution. Yet other equations have several solutions and some even have infinitely many solutions.

A linear equation (in one variable) has the form $ax + b = c$.

Example 13: Solve $7x + 5 = 9$.
Solution: Subtract $5$ from both sides, giving $7x = 4$. Divide both sides by $7$, giving $x = 4/7$.

§1.5. Fractions
A fraction is an expression of the form $\frac{m}{n}$. We sometimes write it as $m/n$. The number on the top is called the numerator and the number on the bottom is called the denominator. When the numerator and denominator are integers, we call the fraction a rational number.

Adding and subtracting fractions, whether or simply arithmetic, can be quite difficult, though, if the denominators are the same it is very easy.

Example 14: $\frac{3}{17} + \frac{4}{17} = \frac{7}{17}$.
Adding seventeenths is no more difficult than adding apples. Three plus four more is seven of them.

Example 15: $\frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} = \frac{x + 1}{x^2 + 1}$.
$\frac{x^2}{x^2 + 1} + \frac{1}{x^2 + 1} = \frac{x^2 + 1}{x^2 + 1} = 1$.
In this last case we were able to simplify the fraction by dividing the numerator and denominator by the same expression.

When the denominators are different we have to find a common multiple, preferably the least common multiple, though that is not absolutely necessary.

Example 16: Simplify $\frac{2}{15} - \frac{3}{20}$.
Solution: Here the least common multiple of the denominators is 60.
$\frac{2}{15} - \frac{3}{20} = \frac{8}{60} - \frac{9}{60} = -\frac{1}{60}$.
However, if we simply used the product of the denominators, 300, we would eventually get the same answer.
$\frac{2}{15} - \frac{3}{20} = \frac{40}{300} - \frac{45}{300} = -\frac{5}{300} = -\frac{1}{60}$. 

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Example 17: Simplify \( \frac{x}{x^2 - 1} - \frac{1}{x^2 + 2x + 1} \).

**Solution:** \( x^2 - 1 = (x - 1)(x + 1) \) and \( x^2 + 2x + 1 = (x + 1)^2 \). Hence both divide \( (x^2 - 1)(x + 1) \).

So \( \frac{x}{x^2 - 1} - \frac{1}{x^2 + 2x + 1} = \frac{x(x + 1)}{(x^2 - 1)(x + 1)} - \frac{x - 1}{(x^2 - 1)(x + 1)} = \frac{x^2 + 1}{(x^2 - 1)(x + 1)} \).

Multiplying fractions is much easier than adding or subtracting them. You simply multiply the numerators and multiply the denominators. And to divide fractions you “invert and multiply”. The fraction that is being divided by has its numerator and denominator swapped to form its multiplicative inverse.

Example 18: Simplify \( \frac{18}{35} \times \frac{20}{27} \).

\[ \frac{18}{35} \times \frac{20}{27} = \frac{360}{945} = \frac{8}{21} \] after “cancelling down” by 45.

But what is much simpler is to do the cancelling before the final multiplications.

\[ \frac{18}{35} \times \frac{20}{27} = \frac{2}{35} \times \frac{20}{3} = \frac{2}{7} \times \frac{4}{3} = \frac{8}{21} . \]

It helps, in algebraic cases, to factorise the numerators and denominators first.

Example 19: Simplify \( \frac{x^2 + 1}{x^2 + 1} \times \frac{x + 1}{x^3 + x} \).

**Solution:** \( x^3 + 1 = (x + 1)(x^2 - x + 1) \) and \( x^3 + x = x(x + 1) \).

\[ \frac{x^2 + 1}{x^2 + 1} \times \frac{x + 1}{x^3 + x} = \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} \times \frac{x + 1}{x(x^2 + 1)} . \]

\[ = \frac{1}{(x + 1)(x^2 - x + 1)} \times \frac{x + 1}{x} = \frac{1}{x(x^2 - x + 1)} . \]

Example 20: Simplify \( \frac{18}{55} \div \frac{16}{25} \).

**Solution:**

\[ \frac{18}{55} \div \frac{16}{25} = \frac{18}{55} \times \frac{25}{16} = \frac{9}{11} \times \frac{5}{8} = \frac{45}{88} . \]

Example 21: Simplify \( \frac{x^4 - 1}{x^4 + 1} \div \frac{x^2 + 1}{x + 1} \).

**Solution:**

\[ \frac{x^4 - 1}{x^4 + 1} \div \frac{x^2 + 1}{x + 1} = \frac{x^4 - 1}{x^4 + 1} \times \frac{x + 1}{x^2 + 1} = \frac{(x - 1)(x + 1)(x^2 + 1)}{x^4 + 1} \times \frac{x + 1}{x^2 + 1} . \]
\[
(x - 1)(x + 1)^2 \quad \frac{x^3 + 1}{x^3 + 1}.
\]