4. Existence and uniqueness.

We rewrite the initial value problem

\[ \dot{x} = f(t, x), \ x(t_0) = x_0 \]  

as an integral equation as follows.

\[ x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s)) ds \]

\[ x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s)) ds \]  

This integral equation is equivalent to the initial value problem (1) and thus when we construct its solution, it will also solve the initial value problem. We will construct a sequence of approximations to the solution of (2) and show that this sequence converges to a solution. The idea is very simple.

We make an initial choice of approximate solution and the natural choice is \( x_0(t) = x_0 \), a constant function. This of course will not satisfy (2), but will provide a second approximation \( x_1(t) \) by

\[ x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s)) ds \]

where all terms on the right hand side are known. Continuing this process, we define the \((n + 1)\)-st approximation \( x_{n+1}(t) \) by

\[ x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s)) ds, \ n = 0, 1, 2,... \]  

The sequence \( \{x_n(t)\}, \ n = 0, 1, 2,... \), is called a sequence of successive approximations to the solution of (3).

**Theorem. (Fundamental existence and uniqueness result).**

Let \( f(t, x) \) and \( \frac{\partial f}{\partial x}(t, x) \) be continuous in a rectangular region \( D = \{(t, x) ; |t - t_0| \leq a, |x - x_0| \leq b\} \), where \( a, b \) are real nonnegative numbers. Then there exists a unique solution \( x = \phi(t) \) of the initial value problem \( \dot{x} = f(t, x), x(t_0) = x_0 \) on some interval \( |t - t_0| \leq h \leq a \).

We prove this result after some preliminary discussions. Since \( f, \frac{\partial f}{\partial x} \) are both continuous in \( D \), by the mean value theorem in the \( x \) variable,

\[ f(t, x_1) - f(t, x_2) = \frac{\partial f}{\partial x} (t, x^*)(x_1 - x_2) \]
4. Existence and uniqueness.

for \((t, x_1), (t, x_2) \in D\), where \(|x^* - x_0| \leq b\). Since \(\frac{\partial f}{\partial x}\) is continuous in the bounded region \(D\), it is bounded in \(D\). That is,

\[
\left| \frac{\partial f}{\partial x} (t, x) \right| \leq K
\]  

(5)

for each \((t, x) \in D\) and some constant \(K \geq 0\). Therefore

\[
|f(t, x_1) - f(t, x_2)| = \left| \frac{\partial f}{\partial x} (t, x^*)(x_1 - x_2) \right| \leq K |x_1 - x_2|
\]  

(6)

for \((t, x_1), (t, x_2) \in D\). This is called a Lipschitz condition for \(f(t, x)\).

Since \(f(t, x)\) is continuous in the bounded region \(D\), it is bounded in \(D\). That is

\[
|f(t, x)| \leq M
\]  

(7)

for each \((t, x) \in D\), and some constant \(M \geq 0\). In fact, we can choose

\[
M = \max_D |f(t, x)|, \quad K = \max_D \left| \frac{\partial f}{\partial x} (t, x) \right|
\]

Let

\[
h = \min \left( a, \frac{b}{M} \right)
\]  

(8)

and define the subset of \(D\),

\[
S = \{ (t, x) : |t - t_0| \leq h, |x - x_0| \leq b \}.
\]

We consider two cases;

(a) \(h = a\). Then \(S = D\).

(b) \(h = \frac{b}{M} < a\). Then \(S \subset D\).

The definitions of \(h\) and \(S\) ensure that any solution \(x = \phi(t)\), to the integral equation (2), satisfies \((t, \phi(t)) \in S\), since for \(|t - t_0| \leq h\),

\[
|\phi(t) - x_0| = \left| \int_{t_0}^{t} f(s, \phi(s)) ds \right|
\]

\[
\leq \int_{t_0}^{t} |f(s, \phi(s))| ds \leq M |t - t_0|
\]

\[
\leq Mh = b.
\]

Each successive approximation, \(x_n(t), n = 0, 1, 2, \ldots\), satisfies the same condition.
4. Existence and uniqueness.

**Lemma 1.** If \(|t - t_0| \leq h\) and \(\{x_n(t)\}, n = 0, 1, 2, \ldots\), is a sequence of successive approximations defined by (3), then

\[ |x_n(t) - x_0| \leq b, \quad n = 0, 1, 2, \ldots . \]

**Proof.** (By induction). The result is trivially true for \(n = 0\). Suppose it is true for any \(n \geq 0\). Then \((t, x_n(t)) \in S \subseteq D\) and hence \(|f(t, x_n(t))| \leq M\) for \(|t - t_0| \leq h\). Therefore

\[
|x_{n+1}(t) - x_0| = \left| \int_{t_0}^t f(s, x_n(s)) ds \right| \\
\leq \int_{t_0}^t |f(s, x_n(s))| ds \leq M|t - t_0| \leq Mh \leq b.
\]

QED.

**Lemma 2.** If \(|t - t_0| \leq h\) then

\[ |f(t, x_{n+1}(t)) - f(t, x_n(t))| \leq K|x_{n+1}(t) - x_n(t)|. \]

**Proof.** This follows from lemma 1 and the Lipschitz condition for \(f\).

QED.

**Proof of the fundamental existence and uniqueness theorem.**

Let \(t_0 \leq t \leq t_0 + h\) (the case \(t_0 - h \leq x \leq t_0\) is treated similarly).

From lemma 1,

\[ |x_1(t) - x_0| \leq M|t - t_0|. \]

Also \(|x_n(t) - x_{n-1}(t)| \leq MK^{n-1} \frac{|t - t_0|^n}{n!}, n = 1, 2, \ldots\)

This we prove by induction. By lemma 1, it is true for \(n = 1\).

Assume that it is true for some \(n = k - 1\). That is, \(|x_{k-1}(t) - x_{k-2}(t)| \leq MK^{k-2} \frac{|t - t_0|^{k-1}}{(k-1)!} .\)

Then from (3),

\[
|x_k(t) - x_{k-1}(t)| = \left| \int_{t_0}^t \{f(s, x_{k-1}(s)) - f(s, x_{k-2}(s))\} ds \right| \\
\leq \int_{t_0}^t |f(s, x_{k-1}(s)) - f(s, x_{k-2}(s))| ds \\
\leq K \int_{t_0}^t |x_k(s) - x_{k-1}(s)| ds \quad \text{(from lemma 2)}
\]

QED.
4. Existence and uniqueness.

\[ \leq K \int_{t_0}^{t} MK^{k-2} \frac{|s - t_0|^{k-1}}{(k-1)!} ds = MK^{k-1} \frac{|t - t_0|^k}{k!} \]

and the result holds for \( n = k \). By induction, since the result holds for \( n = 0 \), it holds for all \( n \). The case \( t_0 - h \leq t \leq t_0 \) is treated similarly.

Comparing the infinite series,

\[ \sum_{n=1}^{\infty} \left[ x_n(t) - x_{n-1}(t) \right], \quad \sum_{n=1}^{\infty} MK^{n-1} \frac{h^n}{n!}, \]

the second series converges to \( \frac{M}{K} \left( e^{Kh} - 1 \right) \) and each term dominates the corresponding term in the first series. By the comparison test, the series

\[ \sum_{n=1}^{\infty} \left[ x_n(t) - x_{n-1}(t) \right] \]

converges absolutely and uniformly on the interval \( |t - t_0| \leq h \).

The \( k \)-th partial sum \( \sum_{n=1}^{k} \left[ x_n(t) - x_{n-1}(t) \right] = x_k(t) - x_0 \) converges, hence \( x_k(t) \) converges absolutely and uniformly on \( |t - t_0| \leq h \). The limit we denote by

\[ \phi(t) = \lim_{k \to \infty} x_k(t) \]

and is a continuous function (a uniform limit of a sequence of continuous function is continuous). Also we can interchange the order of taking limits and integration for such sequences. Therefore

\[ \phi(t) = \lim_{k \to \infty} x_k(t) \]

\[ = x_0 + \lim_{k \to \infty} \int_{t_0}^{t} f(s, x_{k-1}(s)) ds \]

\[ = x_0 + \int_{t_0}^{t} \lim_{k \to \infty} f(s, x_{k-1}(s)) ds \]

\[ = x_0 + \int_{t_0}^{t} f(s, \lim_{k \to \infty} x_{k-1}(s)) ds \]

\[ = x_0 + \int_{t_0}^{t} f(s, \phi(s)) ds, \quad |t - t_0| \leq h. \] (9)
4. Existence and uniqueness.

Thus \( \phi(t) = \lim_{k \to \infty} x_k(t) \) is a solution of the initial value problem. To show that this solution is unique, suppose that \( x = \psi(t) \) is another solution for \( |t - t_0| \leq h \). Then \( \psi(t) = x_0 + \int_{t_0}^{t} f(s, \psi(s)) \, ds \), \( |t - t_0| \leq h \). Subtracting (2) and (9),

\[
|\phi(t) - \psi(t)| = \left| \int_{t_0}^{t} (f(s, \phi(s)) - f(s, \psi(s))) \, ds \right|
\leq K \int_{t_0}^{t} |\psi(s) - \phi(s)| \, ds.
\]

Defining \( z(t) = \int_{t_0}^{t} |\phi(s) - \psi(s)| \, ds \), the above inequality is \( \dot{z}(t) \leq Kz(t) \). Multiplying by \( e^{-K(t-t_0)} \),

\[
e^{-K(t-t_0)}(\dot{z}(t) - Kz(t)) \leq 0.
\]

Integrating,

\[
e^{-K(t-t_0)}z(t) - z(t_0) \leq 0,
\]

\( z(t) \leq e^{K(t-t_0)}z(t_0) \).

But \( z(t_0) = 0 \) by definition, therefore \( z(t) = \int_{t_0}^{t} |\phi(s) - \psi(s)| \, ds \leq 0 \), for \( |t - t_0| \leq h \), which implies that \( |\phi(t) - \psi(t)| = 0 \) or \( \phi(t) = \psi(t) \), \( |t - t_0| \leq h \). The solution is therefore unique.

QED.

**Example 1.** Consider the initial value problem \( \dot{x} = 1 + x^2 \), \( x(0) = 0 \). Then \( f(t, x) = 1 + x^2 \). \( \frac{\partial f}{\partial x}(t, x) = 2x \) are continuous in \( \mathbb{R}^2 \). Let \( D = \{(t, x); |t| \leq a, |x| \leq b\} \) for some \( a, b \).

Then let \( M = \max_{D} |f(t, x)| = \max_{D} |1 + x^2| = 1 + b^2 \). A unique solution exists on the interval \( |t| \leq h = \min \left( a; \frac{b}{1 + b^2} \right) \). We can choose \( a, b \) so as to maximize the value of \( h \). Since the maximum value of \( \frac{b}{1 + b^2} \) is \( \frac{1}{2} \) and occurs for \( b = 1 \) by simple calculus, by choosing \( b = 1, a \geq \frac{1}{2} \), the maximum value of \( h \) is \( \frac{1}{2} \). The fundamental theorem guarantees a unique solution of this initial value problem for \( |t| \leq \frac{\pi}{2} \). This initial value problem is easily solved by separating variables and the unique solution is \( x = \tan t, |t| \leq \frac{\pi}{2} \).

**Example 2.** Consider the initial value problem \( \dot{x} = \sqrt{1 - x^2} \), \( x(0) = 1 \).
4. Existence and uniqueness.

Then \( f(t, x) = \sqrt{1 - x^2} \), \( \frac{\partial f}{\partial t}(t, x) = \frac{-tx}{\sqrt{1 - x^2}} \) are both continuous for \(|t| < 1\). Every region \( D = \{(t, x) ; |t| \leq a, |x - 1| \leq b\} \) for \( a > 0, b > 0 \), contains points where \( f \) is undefined or \( \frac{\partial f}{\partial x}(t, x) \) is not continuous. The hypotheses of the fundamental theorem are not satisfied and we cannot conclude existence of a unique solution in this example. Again by separating variables we can calculate a solution \( x = \sin \left( \frac{t^2 + x}{2} \right) \). However \( x = 1 \) is also a solution and therefore this initial value problem does not have a unique solution.