Liouville's Theorem: If α is real and algebraic of degree d, then there is a positive constant C, depending only on α, such that for every rational p/q,

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^d}$$

Proof. Let f in $\mathbf{Z}[x]$ be such that $f(\alpha) = 0$. Let $f(x) = a_0x^d + a_1x^{d-1} + \ldots + a_d$. Then $f(p/q) \neq 0$, so $q^d f(p/q)$ is a non-zero integer, so

$$|q^d f(p/q)| \geq 1 \tag{1}$$

By the Mean Value Theorem,

$$\frac{f(p/q)|}{|\alpha - (p/q)|} \leq \max |f'(\xi)| \tag{2}$$

where the maximum is over all ξ between α and p/q. If we assume $|\alpha - (p/q)| < 1$ then we can replace the right side of (2) with M, the maximum value of $|f'(x)|$ for $\alpha - 1 \leq x \leq \alpha + 1$. Note that M depends only on α. Comparing with (1) we get

$$\left|\alpha - \frac{p}{q}\right| > \frac{1/M}{q^d}$$

Corollary: $\alpha = \sum_{n=1}^{\infty} 10^{-n!}$ is transcendental.

Proof. Given any $\epsilon > 0$, and any natural number d, choose a natural number k such that $k \geq d$ and $k! > \log_{10}(2/\epsilon)$. Let $\sum_{n=1}^{k} 10^{-n!} = p/q$, so $q = 10^{k!}$, and

$$|\alpha - (p/q)| < 2 \times 10^{-(k+1)!}$$

Then $10^{(k+1)!}/10^{dk!} = 10^{(k+1-d)k!} \geq 10^{k!} > 2/\epsilon$, so

$$|\alpha - (p/q)| < 2 \times 10^{-(k+1)!} < \epsilon 10^{-dk!} = \epsilon/q^d$$

Since ϵ is arbitrary, Liouville's Theorem says α can't be of degree d. Since d was arbitrary, α can't be algebraic.

Cultural enrichment: The algebraic numbers form a countably infinite set, the transcendentals, an uncountably infinite set. The Liouville transcendentals (that is, the numbers that can be proved transcendental using Liouville's Theorem) form an uncountable set of Lebesgue measure zero, the non-Liouville transcendentals form a set of full Lebesgue measure.