A Civilized Three Way Duel – courtesy of Martin Gardner

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Once upon a time, there were three gentlemen who had agreed to a three way duel subject to the following rules. They drew lots to determine who shot first, who shot second, and who shot third, and then this same shooting sequence would repeat until there was only one man left standing. Needless to say, a dead man would lose his turns in the shooting sequence.

Gentleman H is a crack shot, with a 100% record of fatal shots on target. Gentleman E is a moderate shot, with an 80% record of fatal shots on target. Gentleman F is the weakest shot, with only a 50% record of fatal shots on target.

These were three very clever gentlemen who were each able to work out his best strategy, and would adopt this best strategy in the duel. Our problem is to determine which of the three gentlemen had the best chance of survival.

First of all, we must determine the best strategy for each of the three gentlemen. For H and E, the best strategies are clear. Each aims to eliminate the other before turning his attention to F who poses the least threat.

The best strategy for F needs more careful examination.

(1) Suppose that F kills H when it is his turn to shoot. Then he next faces a shot from E with only a 20% chance of survival.

(2) Suppose that F kills E when it is his turn to shoot. Then he next faces a shot from H with certain death.

(3) Suppose that F fails to kill either H or E. Then whoever shoots next will not shoot F, as their best strategies dictate that they aim to eliminate each other first. So if F keeps on missing deliberately by shooting in the air and wait for one of the other two to eliminate the other, then F himself has the next shot, with a 50% chance of success and so at least a 50% chance of ultimate survival.

In summary, H and E aims to eliminate each other before turning his attention on F, while F keeps missing deliberately until one of the others is dead, and has first shot against the survivor.

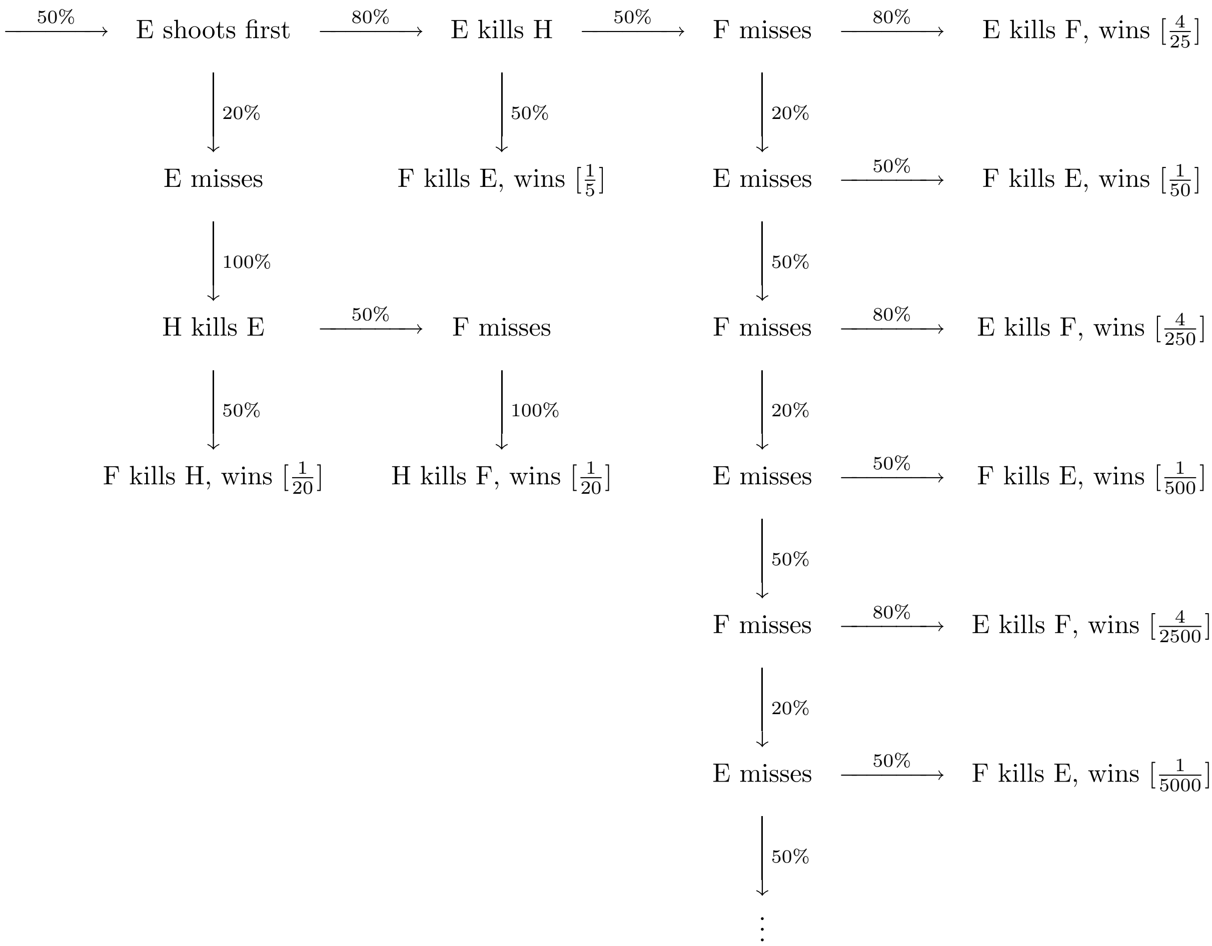
The order of shooting can be any of HEF, HFE, EHF, EFH, FHE, FEH, with equal probability. However, since F is going to miss deliberately until one of his opponents is eliminated, the question is reduced to which of H or E gets the first real shot. There is a 50% chance that H gets to shoot ahead of E, and a 50% chance that E gets to shoot ahead of H.

The case when H gets to shoot ahead of E is summarized by the following finite probability tree:

$$\begin{array}{ccccccc}
\xrightarrow{50\%} & \text{H shoots first} & \xrightarrow{100\%} & \text{H kills E} & \xrightarrow{50\%} & \text{F kills H, wins } [\frac{1}{4}] \\
 & & & \downarrow{\scriptstyle 50\%} & & \\
 & & & \text{F misses} & \xrightarrow{100\%} & \text{H kills F, wins } [\frac{1}{4}]
\end{array}$$

The number in square brackets $[\cdot]$ is the probability of that particular chain of events, and is obtained by multiplying together all the probabilities (fractions) along the path in question.

The case when E gets to shoot ahead of H is more complicated and is summarized by the following infinite probability tree:



Clearly if H is killed first, then there is just a chance that the other two can keep on missing each other for an arbitrarily long time!

If we analyze the two probability trees above, then we can calculate the probability for each gentleman's survival. The probability p_{H} that H survives is given by

$$p_{\mathrm{H}} = \frac{1}{4} + \frac{1}{20} = \frac{3}{10}.$$

The probability p_{E} that E survives is given by

$$p_{\mathrm{E}} = \frac{4}{25} + \frac{4}{250} + \frac{4}{2500} + \ldots = \frac{4}{25}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right) = \frac{4}{25}(1 + 0.111\ldots) = \frac{4}{25}\left(1 + \frac{1}{9}\right) = \frac{4}{25} \times \frac{10}{9} = \frac{8}{45}.$$

The probability p_{F} that F survives is given by

$$p_{\mathrm{F}} = \frac{1}{4} + \frac{1}{20} + \frac{1}{5} + \frac{1}{50} + \frac{1}{500} + \frac{1}{5000} + \ldots = \frac{3}{10} + \frac{1}{5}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right) = \frac{3}{10} + \frac{1}{5} \times \frac{10}{9} = \frac{47}{90}.$$

Hence we arrive at the somewhat counter-intuitive conclusion that

$$p_{\mathrm{F}} > p_{\mathrm{H}} > p_{\mathrm{E}}.$$

In other words, gentleman F, the weakest shot, has the best chance of survival.

Lee Kean has the following comment: We must not expect that in international politics, nations will behave as sensibly as individuals. Fifty-fifty Jones (F), against his own best interests, will blaze away when able at the opponent he imagines to be most dangerous. Even so, he still has the best chance of survival, 44.722 per cent. Brown (E) and Smith (H) find their chances reversed. Eighty-twenty Brown's chances are 31.111 per cent and sure-shot Smith comes in last with 24.167 per cent. Maybe the moral for international politics is even better here.

In this case, F adopts the inferior strategy of trying to eliminate H first and then turn his attention to E. As before, the order of shooting can be any of HEF, HFE, EHF, EFH, FHE, FEH, with equal probability. However, we now have to analyze each of the six cases carefully, and the analysis is rather more complicated. The corresponding probabilities now satisfy

$$p_{\mathrm{F}} = \frac{161}{360}, \qquad p_{\mathrm{E}} = \frac{14}{45}, \qquad p_{\mathrm{H}} = \frac{29}{120}.$$

You may like to analyze this new case. I suggest that you attempt it as a group. It will take quite a bit of time, and you will need A3 size paper or rather small handwriting. The cases HEF and HFE are very simple, while the cases EFH and FEH are the most complicated.